Engineering Electromagnetics
Lab6 – Laplace and Poisson

Objective

The equations of Poisson and Laplace for electrostatics

Maxwell's derivation of Maxwell's equations marked an incredible achievement where a set of equations can completely describe charges and electric current. Gauss's law is one of these equations and it describes electric fields in a vacuum with charge density ρ

(1) $$\nabla \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$$

The electric field E(r) for a given a stationary charge distribution rho(r) can be calculated from

(2) $$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon}\iiint_V \left(\frac{\rho(\vec{r}')\,dV'}{|\vec{r}-\vec{r}'|^2} \right) \hat{r}$$

The electric field E(r) is a vector quantity, therefore, it is often easier to calculate the scalar quantity known as the electrostatic potential From equation 3 rather than calculate the electric field using equation 2.

(3) $$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon}\iiint_V \left(\frac{\rho(\vec{r}')\,dV'}{|\vec{r}-\vec{r}'|} \right)$$

The electric field is the gradient of the potential.

(4) $E = -\nabla\phi$

The electrostatic potential can only be evaluated analytically for the simplest charge configurations. In addition, in many electrostatic problems, conductors are involved and the charge distribution ρ(r) is not known in advance (only the total charge or potential on each conductor is known).

A better approach to determine the electrostatic potential φ is to start with Poisson's equation.

(6) $$\nabla^2\phi = 0$$

Laplace's equation in Cartesian coordinates is written as:

(7) $$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

This is a fundamental equation of electrostatics. It is also an important equation in many branches of physics such as magnetism, gravitation, thermal physics, fluids, and soap bubbles.

Part 1 – Numerical solutions of Poisson's equation and Laplace's equation

We will concentrate only on numerical solutions of Poisson's equation and Laplace's equation. As an introduction, we will only consider [1D] and [2D] cases. The methods discussed can easily be extended to [3D] situations.

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The central difference approximation to the second derivative of the function y(x) is;

(8) $$\frac{d^2y(x)}{dx^2}\Big|_x \approx \frac{y(x+\Delta x)-2y(x)+y(x-\Delta x)}{\Delta x^2}$$

The [2D] space is divided into a grid of *Nx* x *Ny* points with grid spacing $h_x \equiv \Delta x$ in the X direction and $h_y \equiv \Delta y$ in the Y direction.

For computational purposes, it is best to express all quantities defined at the grid positions in terms of grid indices nx and ny where;

$$n_x = 1,2,3,\ldots,N_x \quad \text{and} \quad n_y = 1,2,3,\ldots,N_y$$

Using the central difference approximation of equation 8, we can approximate the value of the potential in terms of the potentials $\phi(n_X, n_Y)$ at the surrounding grid points from Poisson's equation

$$\frac{\phi(n_x+1,n_y)-2\phi(n_x,n_y)+\phi(n_x-1,n_y)}{h_x^{\,2}} + \frac{\phi(n_x,n_y+1)-2\phi(n_x,n_y)+\phi(n_x,n_y-1)}{h_y^{\,2}} = -\frac{\rho}{\varepsilon_0}$$

Rearranging this expression, we get

(9)

$$\phi(n_x,n_y) = \left(\frac{h_x^{\,2}}{2\left(h_x^{\,2}+h_y^{\,2}\right)}\right)\left(\phi(n_x,n_y+1)+\phi(n_x,n_y-1)\right) + \left(\frac{h_y^{\,2}}{2\left(h_x^{\,2}+h_y^{\,2}\right)}\right)\left(\phi(n_x+1,n_y)+\phi(n_x-1,n_y)\right) + \frac{h_x^{\,2}h_y^{\,2}}{2\left(h_x^{\,2}+h_y^{\,2}\right)}\frac{\rho}{\varepsilon_0}$$

The potential at each point is a weighted average of the values of the potential at the surrounding points. So, if we start with known fixed values of the potential on the boundaries, we can repeatedly compute the interior values until we get a convergence in their values. This is known as a **relaxation method** to solve **boundary value problems**.

When using the relaxation method, we mostly specified the boundary conditions and set all values of the potential to zero (or best guess value) at each interior grid. We then repeatedly update all interior values of the potential by the average of the latest neighboring grid point values. We also must test the convergence for the values for the potential. There are many ways in which this can be done. In the Scripts for solving Poisson's equation or Laplace's equation, we will test the convergence by continuing the iterative process while the difference in the sums of the square of the potential at each grid point for the current and previous iterations is greater than specified tolerance.

Section of the Script for the [1D] Laplace's equation showing the iterative process

```
tol = 10;
% dSum Difference in sum of squares / n number of iterations
dSum = 100; n = 0;
```

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```
while dSum > tol
sum1 = sum(sum(V.^2));
for nx = 2: Nx-1
V(nx) = 0.5*(V(nx+1) + V(nx-1));
end
sum2 = sum(sum(V.^2));
dSum = abs(sum2 - sum1);
n = n+1;
end
```

From now on the electrostatic potential is given by the letter V rather than the Greek letter φ.

Problem 1

Solve the [1D] Laplace's equation for the potential with the boundary conditions, *x*=0m; *V* =100V and *x*=5.0m; *V*= 0V, and calculate the electric field in this region.

Download the mscript **cemLaplace03.m**. Review the code so that you understand each line and how Laplace's equation was solved using the relaxation method.

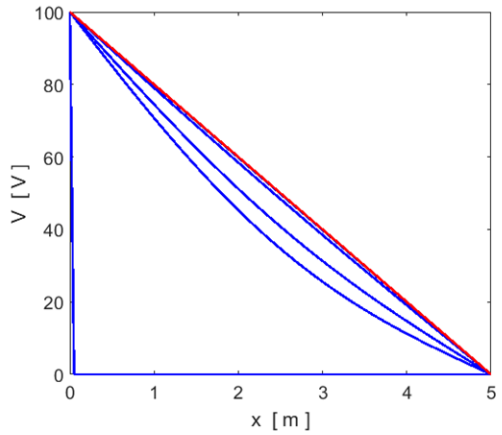
The potential is given by

$$V(n_x) = \frac{V(n_x+1) + V(n_x-1)}{2}$$

where $n_x = 2,3,4,\cdots,N_x-1 \quad V(1) = 100 \quad V(N_x) = 0$

The exact solution of Laplace's equation with the given boundary conditions is V(x)=-20x+100.

The Script **cemLaplace03.m** is used to solve this problem. The exact solution can be compared with the solution using the relaxation method as shown in the figure below. The figure clearly shows that the smaller the value of the tolerance, the better the agreement between the exact solution and the solution using the relaxation method. For a tolerance of tol = 1, the difference between the exact solution and approximate solution is about 0.1%. You always need to be careful in using approximation methods to ensure that the computed values are accurate. You should always check that you have used a smaller enough grid spacing and/or used a smaller enough tolerance.



The electric field is given by $E = -\nabla V$. The electric field is uniform and points in the +X direction and its exact value is 20 V/m. The numerical estimate of the electric field is calculated from the potential using the function gradient, E = -gradient(V,hx);

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The solution to Laplace's equation in a volume is uniquely determined if the potential *V* is specified on the boundary surface *S* of the volume.

Problem 2

There are two parallel semi-infinite metal plates in the XZ plane. One plate is located at $y = -1$m and extends from $x = 0$m to $x = 4$m and the other plate is located at $y = +1$ m and extends from $x = 0$ m to $x = 4$ m. At $x = 0$ m, an insulated strip connects the two semi-infinite plates and is held at a constant potential 100 V. Find the potential and the electric field in the region bounded by the plates.

The configuration is independent of *z,* so this reduces to a [2D] problem. We can find the potential in the interior region by solving Laplace's equation. The region of interest is divided into a rectangular grid of Nx*Ny grid points. The potential at all grid points is set to 0. Then the boundary condition, V=100V is applied to all grid points with y=0 . Then the iterative process updates the potential at all interior grid points until the values of the potential satisfy the tolerance condition. Download and inspect the mscript **cemLaplace02.m**. Make sure you understand each line of the code and how the Laplace's equation is solved using the relaxation method.

The electric field **E** is found by finding the gradient of the potential.

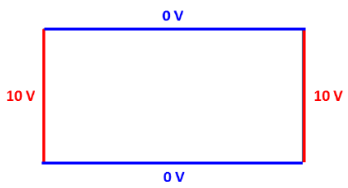
$$E = -\nabla V$$

The Matlab code for the computing the components and magnitude of the electric field is;

```
% electric field
[Exx, Eyy] = gradient(V,hx,hy);
Exx = -Exx; Eyy = -Eyy;
E = sqrt(Exx.^2 + Eyy.^2);
```

Problem 3

Use the mscript **cemLaplace01.m** to investigate the relaxation method to solve Laplace's equation in more detail. Determine the potential V(x, y) inside a rectangle of dimensions 2.0 m x 1.0 m. Set all the boundaries to 10.0 V and all interior grid points to 0 V. Before you do the computation, guess the exact forms of the potential and the electric field. Use the values Nx=7 and Ny=7 for the number of grid points. You can view the number of iterations and the potential in the Command Window after running the program. How many iterations are required to achieve an accuracy of better than 1%? How does the number of iterations change if the number of grid points and are increased?



Problem 4

Adjust problem 3 to fit the following potentials on its Boundaries and solve for the potential distribution.

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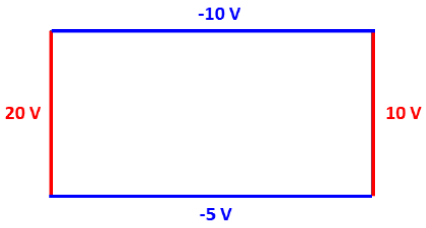

Problem 5

Consider a [2D] charge free region of space with linearly increasing potentials on its boundaries.

You need to modify the mscript (cemLaplace02) for these boundary conditions and uncomment the code in the SETUP section which defines the linearly increasing potentials.

```
% linearly increasing potentials on boundaries
% m1 = 20/maxY; m2 = 20/maxX; m3 = 30/maxY; m4 = 10/maxX;
% b1 = 0; b2 = 20; b3 = 10; b4 = 0;
%
% V(:,1) = m1 .* y + b1;
% V(:,end) = m3 .* y + b3;
% V(end,:) = m2 .* x + b2;
% V(1,:) = m4 .* x + b4;
```