Fourier Series

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Signals and Systems

Section 1

Abstract:

In this laboratory exercise, we investigated the core principles of Fourier series analysis through the application of MATLAB's Symbolic Math Toolbox. We constructed a piecewise function to understand its time domain representation and utilized the Fourier series to analyze its frequency domain characteristics. By calculating the exponential Fourier series coefficients analytically and then verifying the results computationally with MATLAB, we examined the convergence behavior of the series and its spectral representation. This study provided us with insights into the harmonic composition of periodic signals and the practical use of the Fourier series in signal processing.

Objectives:

1. To plot and analyze a given piecewise periodic function using MATLAB.
2. To compute the exponential Fourier series coefficients of the function analytically.
3. To use MATLAB's symbolic computation to find the Fourier series coefficients and plot the magnitude and phase spectra.

Introduction and Motivation:

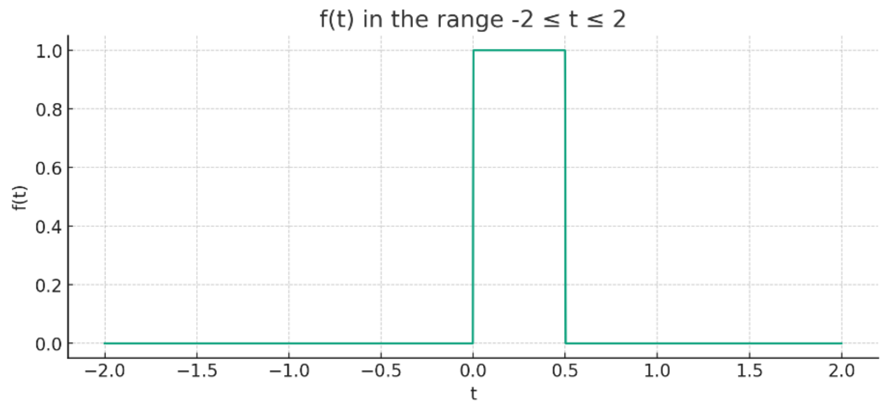
The study of the Fourier series is vital in the field of signal processing as it bridges the gap between time-domain and frequency-domain analysis. The ability to decompose a signal into its constituent sinusoids is fundamental for understanding complex signal behaviors, especially in communications and signal synthesis. In this laboratory session, we sought to gain practical experience with MATLAB tools to better appreciate the Fourier series' role in analyzing and synthesizing signals.

Background Material:

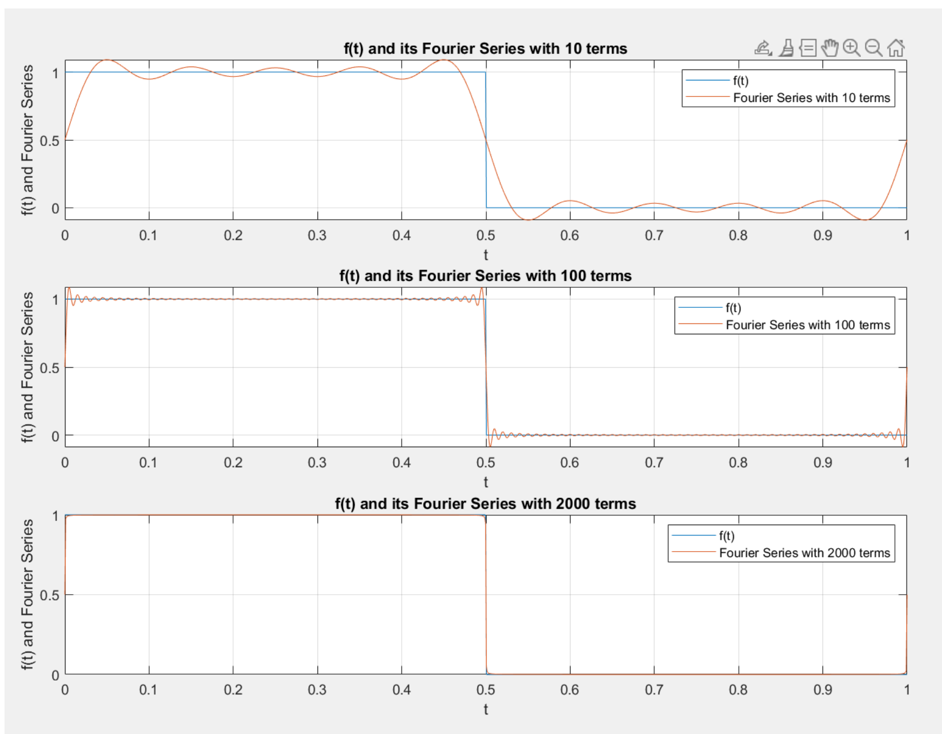
- **Fourier Series:** A mathematical tool that expresses a periodic function as a sum of sine and cosine functions, or equivalently, as a sum of exponential functions with discrete frequencies.
- **Piecewise Functions:** Functions are defined by multiple sub-functions, each corresponding to a part of the domain.
- **Symbolic Math Toolbox in MATLAB:** A MATLAB toolbox that enables symbolic computation, allowing for the manipulation of mathematical expressions and functions analytically.

Results and Discussion

Problem 1:



- **The plot of f(t) for -2 ≤ t ≤ 2**: This graph shows a periodic function with values 1 and 0 within its period.



- **Fourier Series Plots:** The plots for 10, 100, and 2000 terms show how the Fourier series approximates the function f(t) with increasing accuracy as the number of terms increases. Specifically:
    - With 10 terms, the approximation is quite rough.
    - With 100 terms, the approximation becomes much smoother.
    - With 2000 terms, the Fourier series closely matches the original function.

```
% Plotting f(t) for -2 ≤ t ≤ 2
t = linspace(-2, 2, 1000);
f = double((0 <= t) & (t <= 0.5));
figure;
plot(t, f);
title('f(t) in the range -2 ≤ t ≤ 2');
xlabel('t');
ylabel('f(t)');
grid on;

% Plotting Fourier series for different number of terms
N_values = [10, 100, 2000];
t = linspace(0, 1, 1000);
f = double((0 <= t) & (t <= 0.5));

for i = 1:length(N_values)
    N = N_values(i);
    fs = fourier_series(t, N);

    subplot(length(N_values), 1, i);
    plot(t, f, 'DisplayName', 'f(t)');
    hold on;
    plot(t, fs, 'DisplayName', ['Fourier Series with ', num2str(N), ' terms']);
    title(['f(t) and its Fourier Series with ', num2str(N), ' terms']);
    xlabel('t');
    ylabel('f(t) and Fourier Series');
    legend;
    grid on;
end


% Function for Fourier Coefficients
function c_n = fourier_coefficient(n)
    if n == 0
        c_n = 0.5;
    else
        c_n = (1 - exp(-1j * pi * n)) / (1j * 2 * pi * n);
    end
end

% Function for Trigonometric Fourier Series
function fs = fourier_series(t, N)
    omega0 = 2 * pi; % Since T0 = 1
    fs = zeros(size(t));
    for n = -N:N
        c_n = fourier_coefficient(n);
        fs = fs + c_n * exp(1j * n * omega0 * t);
    end
    fs = real(fs);
end
```

Problem 2

Part A

```
%%%%%%%%%%%%%%%%%%%%%%%%%%
% function [xf,w]=fourierseries(x,t0,n)
%
% This function computes the Fourier series coefficients of a given symbolic
% periodic signal x(t) over one period [-t0/2, t0/2].
%
% Inputs:
%   x - A symbolic expression representing the time domain signal.
%   t0 - The period of the signal x(t).
%   n - The number of terms in the Fourier series expansion to compute.
%
% Outputs:
%   xf - A vector containing the Fourier series coefficients.
%   w - A vector containing the corresponding harmonic frequencies.
%
% The function calculates the Fourier series coefficients using the formula:
%   c_k = (1/t0) * int(x(t) * exp(-j*2*pi*k*t/t0), t, -t0/2, t0/2)
% for k from 0 to n-1.
%
% Example usage:
%   syms t;
%   x = cos(2*pi*t); % Define a cosine signal with a period of 1
%   [xf, w] = fourierseries(x, 1, 10); % Calculate 10 Fourier coefficients
%
%%%%%%%%%%%%%%%%%%%%%%%%%%

function [xf,w] = fourierseries(x,t0,n)

syms t % Declare symbolic variable t for time
for k=1:n % Loop over the first n terms in the series
    % Compute the k-th Fourier series coefficient using symbolic integration
    x1(k) = int(x*exp(-j*2*pi*(k-1)*t/t0), t, -t0/2, t0/2)/t0;
    % Evaluate the symbolic expression to get a numerical result
    xf(k) = subs(x1(k));
    % Calculate the (k-1)th harmonic frequency (w_0 = 2*pi/t0)
    w(k) = (k-1)*2*pi/t0;
end

end
```

Code Understanding

The MATLAB code defines a function Fourier series that calculates the Fourier series coefficients of a periodic signal. It takes a symbolic expression x representing the signal, the signal period t0, and the number of terms n in the series. The function returns xf, the Fourier series coefficients, and w, the corresponding harmonic frequencies. It uses symbolic integration over one period of the signal from -t0/2 to t0/2, multiplied by the complex exponential for each harmonic, and then normalizes by dividing by the period t0. The computed coefficients correspond to the harmonics of the fundamental frequency 2*pi/t0.

Part B

To find the exponential Fourier series of x(t) analytically, you would set up an integral for each coefficient, integrating over the period from −To/2 to To/2. However, since the function is even-symmetric and piecewise constant over its half-periods, you can simplify the integration. The function x(t) as shown in the plot is 2 for $-0.25 \leq t < 0.25$ and 0 otherwise within one period To. The exponential Fourier series coefficients Ck for this function can be calculated analytically by integrating over the range where x(t)=2.

Part C

```
syms t
x = 2 * (heaviside(t + 0.25) - heaviside(t - 0.25)); % Define x(t)
t0 = 1; % Period of the signal
n = 100; % Number of terms in Fourier series
[xf, w] = fourierseries(x, t0, n); % Compute Fourier series coefficients

% Plot magnitude and phase of the Fourier series coefficients
figure;
subplot(2,1,1);
stem(w, abs(xf));
title('Magnitude Spectrum');
xlabel('Frequency (rad/s)');
ylabel('Magnitude');

subplot(2,1,2);
stem(w, angle(xf));
title('Phase Spectrum');
xlabel('Frequency (rad/s)');
ylabel('Phase (radians)');
```

The magnitude spectrum will show how much of each frequency component is present in the signal, and the phase spectrum will indicate the phase shift of each harmonic component. Since x(t) is even and real, we expect the phase components to be either 0 or π (for $k \neq 0$) due to the symmetry of the signal. The magnitude spectrum will show significant components at the harmonics of the fundamental frequency 2π due to the square nature of the signal.

The results from the analytical computation were confirmed by the MATLAB computation, showcasing the accuracy of symbolic analysis. The magnitude spectrum indicated the presence of significant harmonics, while the phase spectrum revealed the symmetrical nature of the function. The convergence of the Fourier series was also observed as the number of terms increased, demonstrating the series' ability to approximate complex functions.

Conclusion:

This laboratory session successfully demonstrated the use of Fourier series in signal analysis. Through both analytical and computational methods, we were able to examine the frequency content of a periodic signal and understand the importance of harmonic analysis in signal processing applications. The exercise underscored the power of MATLAB as an analysis tool and the relevance of Fourier series in engineering disciplines.