Modern Guidance

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Missile Guidance Laws

Classic

Non-Homing

Position / orientation of interceptor relative to natural landmarks (stars, etc.) are used to compute guidance commands. Note that the intercept point is a point that can always be described relative to natural landmarks such as celestial bodies, terrain, etc.

Intuitive

Simple guidance algorithms designed to drive the missile to intercept based upon common sense and/or maritime experience, etc.

Modern

Optimal Control

Optimal control guidance laws consider optimizing a cost (final interceptor speed or miss distance) while often considering additional constraints to the optimization problem.

Differential Games

Differential game theory considers an intelligent target which is trying to avoid the interceptor. This results in a two-side optimal control problem

Other Branches

Other branches of modern guidance consider multiple hypothesis target models, fuzzy logic in guidance law selection or guidance gain criteria, or applying principles of other scientific research to the guidance problem

Predictive Guidance

Predictive guidance laws consider the target's trajectory to be known. A target's trajectory is considered in the intercept geometry to generate guidance commands

Missile Guidance Laws

Modern
1960s-Present

- Modern guidance laws are developed using mathematical rigor and optimization theory
- Selection of optimization criteria / constraints by the designer means there is no limit to the number of optimal laws

Optimal Control

Proportional Navigation
- True PN
- Pure PN
- Augmented PN

Optimal Guidance
- Optimal Guidance
- Augmented Guidance
- Rendezvous Guidance
- Kappa Guidance
- H_∞ Control

Differential Games

Differential Games
- Pursuit-Evasion
- Zero Sum Games
- Non-zero Sum Games

Predictive guidance and optimal control guidance laws typically go hand-in-hand

Other Branches

Probability
- IMM / MM

Logic
- Fuzzy Logic
- Neural Network
- Bang-Bang

Predictive Guidance

ZEM
Linear target
Ballistic Target
Maneuvering Target
Single/Multiple Hypothesis Targets

Modern Guidance Laws

- Modern guidance laws are designed to perform a given operation in an optimal manner
  - Optimality is always with respect to some condition
  - Optimality may also have constraints
- Example:
  - Function: Synthesis a trajectory from a guidance law
  - Optimality condition: Minimize the total $a_\perp$ (i.e. minimize $\int_0^T a_\perp^2 \, dt$)
  - Constraints: 1. Hit the target
    2. Interceptor must have a flight path angle of -20°
- Specifying **<u>HOW</u>** the interceptor is to hit the target is a very powerful tool
- Geometric limitations to be mitigated
  - Crossing angle (warhead effectiveness, gimbal limitation)
  - RF propagation loss (multipath effects)

Optimization Toolbox

- There are many methods of optimization including
  - Calculus of Variations
  - Iterative methods
  - Steepest Descent
  - Pontryagin's (Maximization) Principle
- One should always (if possible) choose the method best suited for the problem at hand
- A brief description of the calculus of variations method will be shown, and then we will use that method to develop an optimal guidance law

Calculus of Variations I

Optimization Toolbox

- The essence of the calculus of variations is to find a function $y(x)$ such that the following integral is minimized

$$J = \int_a^b F(x, y, y')dx$$

- Without constraints, this problem is solved using the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0$$

  - Where
    - x is the independent variable
    - y is the state variable

- Calculus of variations allow one to add additional constraints to optimization problem
  - "Hit the target"
  - "The missile must have a flight path angle of -20° when it collides with the target"
  - "The missile must hit the target with a crossing angle of 0° "

Calculus of Variations II

Optimization Toolbox

- Each constraints require an integrals to be solved

$$C_i = \int_a^b f_i(x, y, y')dx$$

$i = 1, 2, 3, \ldots n^{\text{th}}$ constraint

- Now the Euler-Lagrange equation for optimality becomes

$$\frac{\partial F}{\partial y} - \sum_{i=1}^{n} k_i \frac{\partial f_i}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'} - \sum_{i=1}^{n} k_i \frac{\partial f_i}{\partial y'}\right] = 0$$

  - Where k_i constants are solved by initial or final conditions

Optimal Guidance (OG) to a Predicted Intercept Point (PIP)

- Derive an optimal guidance law to a predicted intercept point
  - Optimality condition: Minimize the induced drag over the flight of the missile
  - Constraints: 1. Hit the target
- Remember from previous lectures:
  - Induced drag, $D_i = C_{D_I}\, Q\, S_{Ref}$
  - $C_{D_I} = \dfrac{n_z^2\, W^2}{(C_{N_\alpha})\, Q^2\, S_{ref}^2} \propto n_z^2$ ⟺ Induced drag is minimized when the square of the acceleration us minimized
- By minimizing the square of the acceleration over the trajectory, induced drag is minimized over the trajectory
  - i.e. $\min \int_0^T n_z^2 \; dt$

Notes on Minimizing Induced Drag

OG to a PIP

- The concept of optimization by minimizing induced drag is a canonical form of guidance optimization
  - It is assumed to be the most efficient trajectory (least amount of missile speed loss)
- To claim minimization of induced drag will maximize intercept velocity is only true if one ignores the contribution of zero-lift drag or assumes the summation of induced drag over time is much larger than the summation of zero-lift drag over time, i.e.

$$\left(\int_0^{T_0} D_i \; dt \;=\int_0^{T_0} C_{D_I}\, Q\, S_{Ref}\; dt\right) \gg \left(\int_0^{T_0} D_{ZLD}\; dt = \int_0^{T_0} C_A\, Q\, S_{Ref}\; dt\right)$$

- A closed form solution to the problem of maximizing missile speed when intercepting a target in the presence of a realistic atmospheric model does not exist today

Optimal Guidance Derivation

OG to a PIP

- The illustration to the right is the same as used for the derivation of midcourse PN
- Similar to the previous derivations, we start by defining the basic geometry of the problem

Eq. OG-1 $$\delta = \gamma_M + \sigma$$

Eq. OG-2 $$z_M - z_f = R\sigma$$

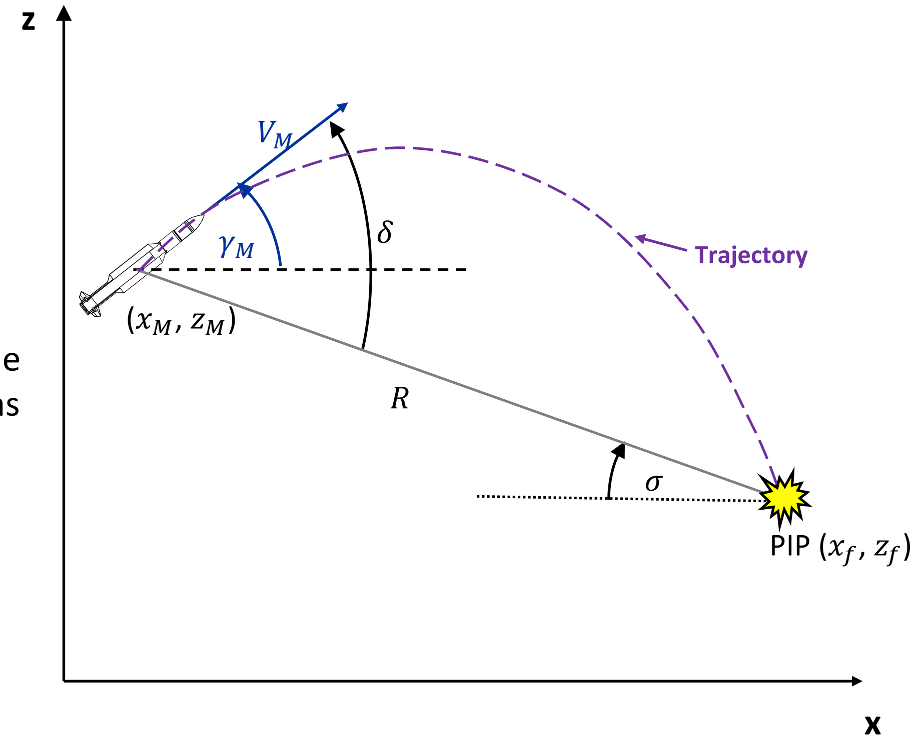
- The relationship of running time to time-to-go has been defined in the previous lecture, as is the transformation of derivatives with respect to T as opposed to t. To recap:

Eq. OG-3 $$T = T_0 - t$$

Eq. OG-4 $$\frac{d}{dT}F = -\frac{d}{dt}F$$

- Small angle approximations allow us to define

Eq. OG-5 $$R = V_M \mathrm{T} \cos(\delta) \cong V_M T$$



Optimal Guidance Derivation II

OG to a PIP

- An attempt is made to describe the acceleration perpendicular to the missile velocity vector in terms of γ_M, σ, and/or δ and their derivatives
- To do this, we take the derivative of the positional error (i.e. $z_M - z_f$) with respect to time, which results in a mixture of current time derivatives and time-to-go, T, due to the approximation made in Eq. OG-5
- Two key assumptions are used to arrive at Eq. OG-8 and Eq. OG-9, as we assume that the intercept point does not move AND the missile speed is constant, but not its direction

Thus, OG-2 and its derivative become

Eq. OG-6 $z_M - z_f = V_M \, T \, \sigma$

Eq. OG-7 $\dot{z}_M - \dot{z}_f = -V_M \, \sigma + V_M \, T \, \dot{\sigma}$

From the description of the problem, the following is true

Eq. OG-8 $\dot{z}_f = 0$

Eq. OG-9 $\dot{z}_M = V_M \gamma_M$

Eq. OG-10 $\dot{\sigma} = \dot{\delta} - \dot{\gamma}_M$

OG-1, and OG-8 through OG-10 can be substituted into OG-7 to yield

Eq. OG-11 $V_M \gamma_M = -V_M \, (\delta - \gamma_M) + V_M \, T(\dot{\delta} - \dot{\gamma}_M)$

Defining the Control

OG to a PIP

- The expressions derived for $a_\perp$ in Eq. OG-12 and Eq. OG-13 are important – each in their own right
- Eq. OG-12 is a basic law of motion for an object moving in a circle. Since the missile speed must remain constant by our problem definition (assumption), Eq. OG-12 will be used as the control to define how the missile will change course
- Eq. OG-13 is a different form of the same parameter, but it is written in terms of a state equation of the system (δ) and its derivative (δ') and the independent variable, T
- One will use Eq. OG-13 in the cost function integral in an attempt to minimize $a_\perp^2$

We note from our previous work that the missile acceleration perpendicular to the velocity vector is described in terms of missile speed and the rate of change of the flight path angle

Eq. OG-12 $$\mathrm{a}_\perp = V_M\dot{\gamma} = -V_M\gamma'$$

Substituting OG-12 in OG-11 gives the following expression for $a_\perp$in terms of t or T

Eq. OG-13 $$\mathrm{a}_\perp = V_M\left(\dot{\delta} - \frac{\delta}{T}\right) = -V_M\left(\delta' + \frac{\delta}{T}\right)$$

We wish to develop a guidance law which will minimize induced drag, which is related to acceleration as

$$\mathrm{D_i} \sim a_\perp^2$$

The Optimization Cost Function

OG to a PIP

Notes

- Induced drag is related to the square of the acceleration
- Describing a cost function which minimizes the integral of $a_\perp^2$ over the flight time will provide the optimal trajectory for this problem
- However, to be successful guidance law, the missile must hit the target. Thus, our first (and only constraint) is for the missile to hit the target
- This is achieved by requiring the final heading to be zero
  - At time-to-go of zero, the heading error must be zero
  - $\delta(T = 0) = 0$

To minimize induced drag, we minimize the square of the acceleration perpendicular to the velocity vector

Eq. OG-14 $J = \int_0^{T_0} a_\perp^2 \; dt$

Substituting OG-13 into OG-14

Eq. OG-15 $J = V_M^2 \int_0^{T_0} \left(\delta' + \frac{\delta}{T}\right)^2 \; dT$

The constraint that requires the missile hit the target must be expressed as an integral. Therefore,

Eq. OG-16 $\delta_f = \delta_0 + \int_0^{T_0} \dot{\delta} \; dt \; = \delta_0 - \int_0^{T_0} \delta' \; dT = 0$

OG-15 and OG-16 are will be used to derive an optimal guidance law using the Calculus of Variations

Calculus of Variations Problem Statement

OG to a PIP

Notes

- To solve the optimization problem, one needs to define the integrand to be optimized and any constraint integrands
- The integrand to be maximized is $F(x, y, y')$, which is the square of the commanded acceleration
- $f_1(x, y, y')$ is the constraint integrand
- The constraint of hitting the target is based on two principles
  - Heading error is zero at intercept
  - Intercept will occur when $T = 0$

OG-15 is in the form required by the Euler-Lagrange equation within the calculus of variations method:

$$J = \int_a^b F(x, y, y')dx = V_M^2 \int_0^{T_0} \left(\delta' + \frac{\delta}{T}\right)^2 dT$$

Eq. OG-17 $F(T, \delta, \delta') = \left(\delta' + \frac{\delta}{T}\right)^2$

OG-16 can be rearranged to match the form required by the constraint equation

$$C_i = \int_a^b f_i(x, y, y')dx \Longrightarrow \delta_0 = \int_0^{T_0} \delta' \; dT$$

Eq. OG-18 $f_1(T, \delta, \delta') = \delta'$

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The Euler-Lagrange Equation

OG to a PIP

- The Euler-Lagrange equation is reprinted below for convenience

$$\frac{\partial F}{\partial y} - \sum_{i=1}^{n} k_i \frac{\partial f_i}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'} - \sum_{i=1}^{n} k_i \frac{\partial f_i}{\partial y'}\right] = 0$$

- For our problem, the first summation term is equal to zero since OG-21 is equal to zero
- The constant, k_1, is the parameter which must be solved using (initial and/or final) known conditions
- The process of solving the Euler-Lagrange begins by taking a number of derivatives which are to be used in the above equation

The partial derivatives required for the Euler-Lagrange equation are as follows:

Eq. OG-19 $$\frac{\partial F}{\partial \delta} = 2\left(\delta' + \frac{\delta}{T}\right)\frac{1}{T}$$

Eq. OG-20 $$\frac{\partial F}{\partial \delta'} = 2\left(\delta' + \frac{\delta}{T}\right)$$

Eq. OG-21 $$\frac{\partial f_1}{\partial\, \delta} = 0$$

Eq. OG-22 $$\frac{\partial f_1}{\partial\, \delta'} = 1$$

Substituting OG-19 through OG-22 into the Euler-Lagrange equation

Eq. OG-23 $$2\left(\delta' + \frac{\delta}{T}\right)\frac{1}{T} = \frac{d}{dT}\left[2\left(\delta' + \frac{\delta}{T}\right) - k_1\right]$$

Solving the Differential Equation

OG to a PIP

- After simplifying Eq. OG-23, one can quickly recognize that Eq. OG-24 can be rewritten in terms of $a_\perp$, making Eq. OG-25 a simple – and solvable - differential equation
- Eq. OG-27 is the answer the differential equation. However, it does not provide a complete answer as the constant of integration ($a_\perp$) still remains as an unknown
- To solve for the unknown, we substitute the Eq. OG-13 into the left ($a_\perp = -V_M\left(\delta' + \frac{\delta}{T}\right)$) side of Eq. OG-27 and multiply both sides of the equation by T

The Euler-Lagrange equation is now simplified

Eq. OG-24 $$\left(\delta' + \frac{\delta}{T}\right)\frac{1}{T} = \frac{d}{dT}\left[\left(\delta' + \frac{\delta}{T}\right)\right]$$

It is recognized that OG-24 can be put in terms of $a_\perp$ using Eq. OG-13

Eq. OG-25 $$\frac{a_\perp}{T} = \frac{d}{dT}[a_\perp]$$

This equation can be rewritten in a form which is easy to solve:

Eq. OG-26 $$\frac{a_\perp}{T} = \frac{d[a_\perp]}{d\,T}$$

Eq. OG-27 $$a_\perp = a_{\perp 0}\frac{T}{T_0}$$

Eq. OG-28 $$-V_M(\delta' T + \delta) = a_{\perp 0}\frac{T^2}{T_0}$$

The Optimal Guidance Law

OG to a PIP

- The step used in Eq. OG-29 and OG-30 is the only part of this derivation that may not be blatantly obvious to the user
- Once the substitution is made in Eq. OG-30, the rest of the derivation is trivial
- One can see in OG-33, that the optimal solution is the same as the proportional navigation guidance law derived earlier with the navigation gain constant set to 3
- Previously, the relationship between flight path angle, $\dot{\gamma}_M$, and line of sight rate, $\dot{\sigma}$, was assumed. However, no such assumption was made in this derivation yet the result it is clear that the proportional relationship exists

One notes the following :

Eq. OG-29 $$\frac{d}{dT}(\delta\, T) = \delta' T + \delta$$

Using the relationship from Eq. OG-29, Eq. OG-28 can be rewritten

Eq. OG-30 $$-V_M \frac{d}{dT}(\delta\, T) = a_{\perp 0} \frac{T^2}{T_0}$$

Eq. OG-31 $$-V_M\, d(\delta\, T) = a_{\perp 0} \frac{T^2}{T_0}\, dT$$

Integrating both sides, and evaluating at $T = T_0$

Eq. OG-32 $$-V_M\, \delta\, T = \frac{a_{\perp 0}}{3} \frac{T^3}{T_0}$$

Eq. OG-33 $$\boldsymbol{a_{\perp 0} = -\frac{3\, V_M\, \delta_0}{T_0}}$$

The Optimal Solution Which Minimizes Induced Drag is Proportional Navigation

PN vs OG and the Value of 3

- One can see that the optimal navigation to an intercept point is the same as proportional navigation to an intercept point with a navigation gain of 3

$$\boldsymbol{a}_{\perp} = -\frac{3\,\boldsymbol{V_M}\,\boldsymbol{\delta_0}}{\boldsymbol{T_0}} \;=\; -\boldsymbol{K}\frac{\boldsymbol{V_M}\boldsymbol{\delta_0}}{\boldsymbol{T_0}} = -\boldsymbol{K}\,\boldsymbol{V_M}\,\dot{\boldsymbol{\sigma}}$$

- What does the selection of a particular navigation gain physically mean?
- To help answer that question, some basic properties of proportional navigation will be discussed
  - δ, $\dot{\sigma}$, and $a_{\perp}$ affected as a function of time-to-go, T
  - Trajectory synthesis

Characteristics of the PN/OG as a Function of TGO (T)

- While it was proven that the optimal value of K to minimize induced drag is 3, guidance system designers will often use other values of K (often to increase missile responsiveness to target maneuvers or overcome time constant response issues)
- Therefore, the characteristics of proportional navigation for any value of K is shown to the right. Note that it was proven that the optimal guidance law is the proportional navigation guidance law with K=3
- The state variable and trajectory parameters over a normalized time period, τ, can be derived from the equations we worked through today (sounds like homework, doesn't it?)

We can normalize the characteristics of the guidance law by using the variable, τ, defined as

$$\tau = \frac{T}{T_0}$$

This provides us a clean, simple way to show the trajectory characteristics as function of τ from 1 (now) to 0 (intercept)

$$\delta = \delta_0\, \tau^{K-1}$$

$$\gamma = \gamma_0 - K\frac{\delta_0}{K-1}\left(1 - \tau^{K-1}\right)$$

$$\dot{\sigma} = \dot{\sigma_0}\, \tau^{K-2}$$

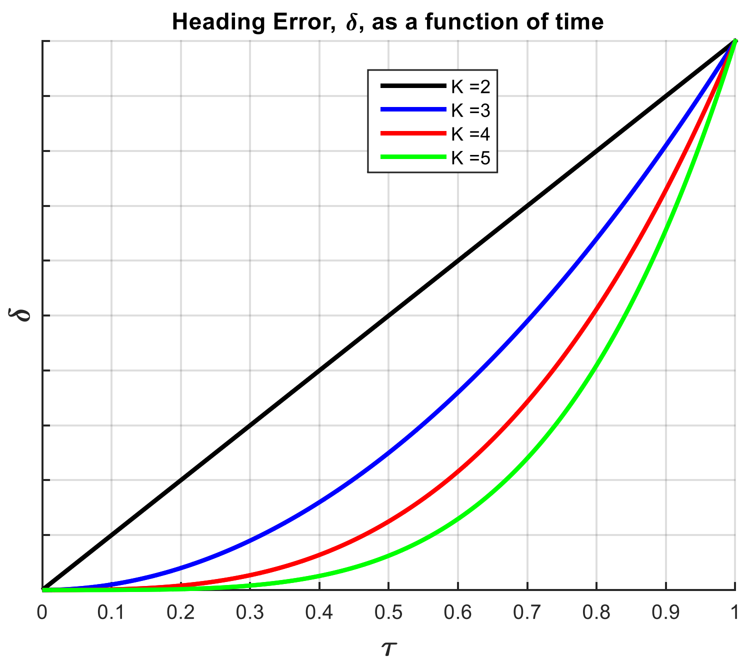
$$\sigma = \sigma_0 + \frac{\delta_0}{K-1}\left(1 - \tau^{K-1}\right)$$

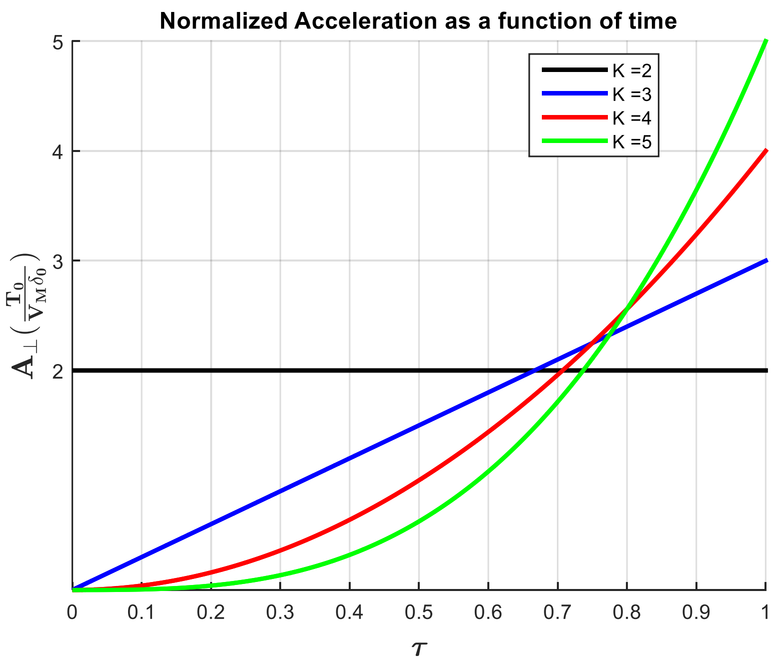
$$z = z_f + R_0\left[\sigma_0\, \tau + \frac{\delta_0}{K-1}\left(\tau - \tau^{K}\right)\right]$$

$$x = x_f - R_0\, \tau$$

δ and $a_\perp$ Over Time

- The state variable (δ) and the control ($a_\perp$) are a function of the normalizing parameter τ raised to a power
  - Acceleration goes as τ^{K-2} $\qquad \delta\,(T) = \delta_0\,\tau^{K-1}$
  - Heading error goes as τ^{K-1} $\qquad a_\perp(T) = a_{\perp 0}\,\tau^{K-2}$





Physical Interpretation of K

- Min value of K to guarantee an intercept is 2
- Increasing K increases the missile's responsiveness to heading error
- K of 3 will minimize the induced drag on the missile
  - Maximizes intercept velocity ***if zero lift drag is negligible with respect to induced drag***
  - "Classic optimization"

K	Result
1	**Increasing acceleration, No intercept**
2	**Constant acceleration, Trajectory is the arc of a circle**
3	**Linearly decreasing acceleration, min induced drag condition**
4	**Exponentially decreasing acceleration**

How Should One Think Of Different Values of K?

- One can arrive at different values of K as optimal solutions to the guidance problem if a slight modification is made to the cost function, J
- Rewrite Eq. OG-15, but in a more generic fashion by introducing an exponential dependency on time-to-go, T^p

$$J = V_M^2 \int_0^{T_0} \frac{\left(\delta' + \frac{\delta}{T}\right)^2}{T^p}\, dT$$

- When $p = 0$, there is no difference between the cost function above and the one in Eq. OG-15
- It can be shown, that the optimal value of K for this revised cost function is simply $K = p + 3$

K Describes a Late Maneuver Penalty in the Cost Function!

OG to a PIP with a Shaping Constraint

- Derive an optimal guidance law to a predicted intercept point
  - Optimality condition: Minimize the induced drag over the flight of the missile
  - Constraints: 1. Hit the target

    2. Have a final flight path angle of γ_f
- The concept of a prescribed flight path angle is important when trying to meet geometric constraints that were discussed in previous lectures
  - Expanding crossrange capability
  - Mitigating multipath
  - Specific approach geometry
- Since this problem is the same as the previous problem, with the additional constraint of a final flight path angle, we can borrow heavily from the previous derivation
  - Eq. OG-1 through Eq. OG-16 are identical and will not be re-derived
  - We will pick up with the cost function and constraint integrals, and we will begin labeling equations with Eq. SG-17 as SG-1 through SG-16 are equal to OG-1 through OG-16

Euler-Lagrange Equation

OG to a PIP with a Shaping Constraint

Notes

- The core of this problem is identical to the first optimization problem
  - The cost function and the cost integrand $F(x, y, y')$ is the same
  - The first constant function and the constraint integrand $f_1(x, y, y')$ is the same
- The optimization problem now has an additional constraint which is a function of the flight path angle, γ (Eq. SG-18b)
  - This constraint has to be described as a function of the state variable, δ, and its derivative (Eq. SG-20)
  - To do this, we equate the following expressions for $a_\perp$
    - $a_\perp = -V_M\,\gamma' = -V_M\left(\delta' + \frac{\delta}{T}\right)$
    - $-\int_0^{T_0}\gamma'\ dT = \frac{1}{V_M}\int_0^{T_0}\left(\delta' + \frac{\delta}{T}\right)\,dT$

OG-17 and OG-18 are repeated here as SG-17 and SG-18 for convenience

$$J = \int_a^b F(x, y, y')dx = V_M^2 \int_0^{T_0} \left(\delta' + \frac{\delta}{T}\right)^2\ dT$$

Eq. SG-17 $F(T, \delta, \delta') = \left(\delta' + \frac{\delta}{T}\right)^2$

Eq. SG-18 $C_i = \int_a^b f_i(x, y, y')dx$

Eq. SG-18a $\delta_f - \delta_0 = -\int_0^{T_0}\delta'\ dT$

Eq. SG-18b $\gamma_f - \gamma_0 = -\int_0^{T_0}\gamma'\ dT$

Eq. SG-19 $f_1(T, \delta, \delta') = \delta'$

Eq. SG-20 $f_2(T, \delta, \delta') = \left(\delta' + \frac{\delta}{T}\right)$

Euler-Lagrange Equation

OG to a PIP with a Shaping Constraint

- The Euler-Lagrange equation is reprinted below for convenience

$$\frac{\partial F}{\partial y} - \sum_{i=1}^{n} k_i \frac{\partial f_i}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'} - \sum_{i=1}^{n} k_i \frac{\partial f_i}{\partial y'}\right] = 0$$

- The constants, k_1 and k_2, are the parameters which must be solved using (initial and/or final) known conditions
- The equation set to the right is identical to the set of equations used during the first derivation, augmented by the two additional equations required to represent the flight path angle constraint (Eq. SG-25 and Eq. SG-26)

The partial derivatives required for the Euler-Lagrange equation are as follows:

Eq. SG-21 $$\frac{\partial F}{\partial \delta} = 2\left(\delta' + \frac{\delta}{T}\right)\frac{1}{T}$$

Eq. SG-22 $$\frac{\partial F}{\partial \delta'} = 2\left(\delta' + \frac{\delta}{T}\right)$$

Eq. SG-23 $$\frac{\partial f_1}{\partial \delta} = 0$$

Eq. SG-24 $$\frac{\partial f_1}{\partial \delta'} = 1$$

Eq. SG-25 $$\frac{\partial f_2}{\partial \delta} = \frac{1}{T}$$

Eq. SG-26 $$\frac{\partial f_2}{\partial \delta'} = 1$$

Solving the Differential Equation

OG to a PIP with a Shaping Constraint

- The Euler-Lagrange Equation can be rewritten in terms of $a_\perp$ and simplified, as is done in Eq. SG-27 through Eq. SG-29
- The "trick" of multiplying both sides of the equation by $\frac{1}{T}$ may seem strange, but it affords one the ability to describe the equation in a more tractable form by recognizing the following relationship

$$\frac{d}{dT}\left[\frac{a_\perp}{T}\right] = \frac{a_\perp'}{T} - \frac{a_\perp}{T^2}$$

- Integration of Eq. SG-31 can be done in steps prior to arriving at Eq. SG-32

$$\mathrm{d}\left[\frac{\mathrm{a}_\perp}{T}\right] = -\frac{k_2}{2}\frac{1}{T^2}\,dT$$

$$\frac{a_\perp}{T} - \frac{a_{\perp 0}}{T_0} = -\frac{k_2}{2}\left(\frac{1}{T} - \frac{1}{T_0}\right)$$

The Euler-Lagrange equation is now

Eq. SG-27 $$2\left(\delta' + \frac{\delta}{T}\right)\frac{1}{T} - \frac{k_2}{T} = \frac{d}{dT}\left[2\left(\delta' + \frac{\delta}{T}\right) - k_1 - k_2\right]$$

Eq. SG-28 $$2(a_\perp)\frac{1}{T} - \frac{k_2}{T} = \frac{d}{dT}\left[2(a_\perp) - k_1 - k_2\right]$$

Eq. SG-29 $$2\frac{a_\perp}{T} - \frac{k_2}{T} = 2\frac{d}{dT}\left[a_\perp\right] = 2\,a_\perp'$$

Both sides of the above equation can be multiplied by $\frac{1}{2T}$ and simplified:

Eq. SG-30 $$\frac{a_\perp'}{T} - \frac{a_\perp}{T^2} = -\frac{k_2}{2}\frac{1}{T^2}$$

Eq. SG-31 $$\frac{d}{dT}\left[\frac{a_\perp}{T}\right] = -\frac{k_2}{2}\frac{1}{T^2}$$

Multiplying both sides by dT and integrating yields:

Eq. SG-32 $$a_\perp = a_{\perp_0}\left(\frac{T}{T_0}\right) + \frac{k_2}{2}\left(1 - \frac{T}{T_0}\right)$$

Solving for Constants $a_{\perp 0}$ and k_2

OG to a PIP with a Shaping Constraint

- Eq. SG-33 is the normalized form of the guidance law, which is normalized using the ratio of current time-to-go to initial time-to-go

$$\tau = \frac{T}{T_0}$$

- Normalization is done for convenience, but it also allows one to synthesize trajectories in a generic form.

- The conversion from dT to $d\tau$ between Eq. SG-35 and Eq. SG-36 results in a factor of T_0^2 being pulled out of the integrand on the right.

$$-\int_{T_0}^{0} a_\perp \; T \; dT = -T_0^2 \int_{1}^{0} a_\perp \; \tau \; d\tau$$

The guidance law can be written in normalized form as

Eq. SG-33 $$a_\perp = a_{\perp_0} \, (\tau) + \frac{k_2}{2} \, (1 - \tau)$$

To utilize the guidance law, the two constants of the above equation ($a_{\perp_0}$ and k_2) must be determined through initial and/or final conditions

We start with the definition of $a_\perp$ and multiply by T

Eq. SG-34 $$-a_\perp = V_M \; T \; \delta' + V_M \; \delta = V_M \frac{d}{dT} [T \; \delta]$$

Integrating Eq. SG-34 yields:

Eq. SG-35 $$V_M \int_{T_0 \; \delta_0}^{0} d(T \; \delta) = -\int_{T_0}^{0} a_\perp \; T \; dT$$

Eq. SG-36 $$-\frac{V_M \delta_0}{T_0} = \int_0^1 a_\perp \; \tau \; d\tau$$

Solving for Constants $a_{\perp 0}$ and k_2

OG to a PIP with a Shaping Constraint

- For simplicity, one can define

$$C_1 = -\frac{V_M \delta_0}{T_0}$$

- The setting of the constant to the temporary variable C_1 will allow for clearer computations as the value of the constants $a_{\perp 0}$ and k_2 is determined

Eq. SG-36 can be solved for by substituting Eq. SG-33 for $a_{\perp}$

Eq. SG-37 $$-\frac{V_M \delta_0}{T_0} = \int_0^1 \left(a_{\perp_0}(\tau) + \frac{k_2}{2}(1-\tau)\right) \tau \, d\tau$$

Through simple integration we arrive at the following

Eq. SG-38 $$C_1 = \frac{a_{\perp 0}}{3} + \frac{k_2}{2}\left(\frac{1}{2} - \frac{1}{3}\right)$$

or

Eq. SG-39 $$3C_1 = a_{\perp 0} + \frac{k_2}{4}$$

Next, a second equation is needed to solve for the two constants. The second equation is procured from the shaping constraint

Solving for Constants $a_{\perp 0}$ and k_2

OG to a PIP with a Shaping Constraint

- For simplicity, one can define

$$C_2 = \frac{V_M\,(\gamma_f - \gamma_0)}{T_0}$$

- The setting of the constant to the temporary variable C_2 will allow for clearer computations as the value of the constants $a_{\perp 0}$ and k_2 is determined

Using the second constraint as a starting point

Eq. SG-40 $$V_M\left(\gamma_f - \gamma_0\right) = \int_0^{T_0} a_\perp \, dT = T_0 \int_0^1 a_\perp \, d\tau$$

Using the substitution of the constant A_2 and the definition of $a_\perp$

Eq. SG-41 $$C_2 = \int_0^1 a_\perp \, d\tau$$

Eq. SG-42 $$C_2 = \int_0^1 \left(a_{\perp_0}\,(\tau) + \frac{k_2}{2}\,(1-\tau)\right) d\tau$$

Once again, integrating provides the second equation required to solve for the constants

Eq. SG-43 $$C_2 = \frac{a_{\perp 0}}{2} + \frac{k_2}{2}\left(1 - \frac{1}{2}\right)$$

Eq. SG-44 $$2\,C_2 = a_{\perp 0} + \frac{k_2}{2}$$

Closed Form Solution for $a_\perp$

OG to a PIP with a Shaping Constraint

- Remember in **Eq. SG-33**, a closed form solution to $a_\perp$ was defined in terms of the constants $a_{\perp 0}$ and k_2. It is repeated here for convenience

$$a_\perp = a_{\perp_0}(\tau) + \frac{k_2}{2}(1-\tau)$$

Eq. SG-39 and Eq. SG-44 describe the two unknowns in terms of constants C_1 and C_2. Solving for the two unknowns

Eq. SG-45 $$\frac{k_2}{2} = -6\,C_1 + 4\,C_2$$

Eq. SG-46 $$a_{\perp_0} = 6\,C_1 - 2\,C_2$$

These two equations can be substituted into Eq. SG-33, and recalling the definitions of C_1 and C_2 provides the closed form solution for the acceleration commands

Eq. SG-47

$$\boldsymbol{a_\perp = \left(-6\frac{V_M\,\delta_0}{T_0} - 2\frac{V_M\,(\gamma_f - \gamma_0)}{T_0}\right)(\tau) + \left(6\frac{V_M\,\delta_0}{T_0} + 4\frac{V_M\,(\gamma_f - \gamma_0)}{T_0}\right)(1-\tau)}$$

Instantaneous $a_\perp$

OG to a PIP with a Shaping Constraint

- At any instant, the commanded acceleration, $a_\perp$, can be found by setting $\tau = 1$

$$a_\perp = -6\frac{V_M\,\delta_0}{T_0} - 2\frac{V_M\left(\gamma_f - \gamma_0\right)}{T_0}$$

- One can see that there are two components to the acceleration command
  - Heading error, δ_0
  - Flight path angle delta, $\gamma_f - \gamma_0$
- At intercept (i.e. $T = 0$), both heading error and flight path angle delta must be zero or the commanded acceleration is infinite

Odds and Ends

OG to a PIP with a Shaping Constraint

- The optimal solution for guidance to a PIP with shaping constrain has been defined for a two dimensional problem

$$\boldsymbol{a_\perp = -6\frac{V_M\,\delta_0}{T_0} - 2\frac{V_M\,(\gamma_f - \gamma_0)}{T_0}}$$

- There are two planes in which the acceleration commands must be generated which are defined uniquely
  - δ_0 is defined in the plane along the missile to intercept point vector
  - $\gamma_f - \gamma_0$ is defined by the unit vector γ_f and the unit vector of V_M
- For the general 3-D case, one must be careful during this derivation as the angular relationship of $\sigma = \delta - \gamma$ is no longer true unless both δ and $\gamma_f - \gamma_0$ are defined to be in the sample plane

$$\boldsymbol{\sigma \neq \delta - \gamma}$$

Characteristics OG to a PIP with a Shaping Constraint as a Function of τ

- The state variables, δ and γ, and the position values, x and z, are defined as follows:

$$\delta = \delta_0\,(4\tau^2 - 3\tau) + 2(\gamma_f - \gamma_0)(\tau^2 - \tau)$$

$$\gamma = \gamma_0 - 6\,\delta_0\tau\,(1 - \tau) + (\gamma_f - \gamma_0)(1 - 3\tau)(1 - \tau)$$

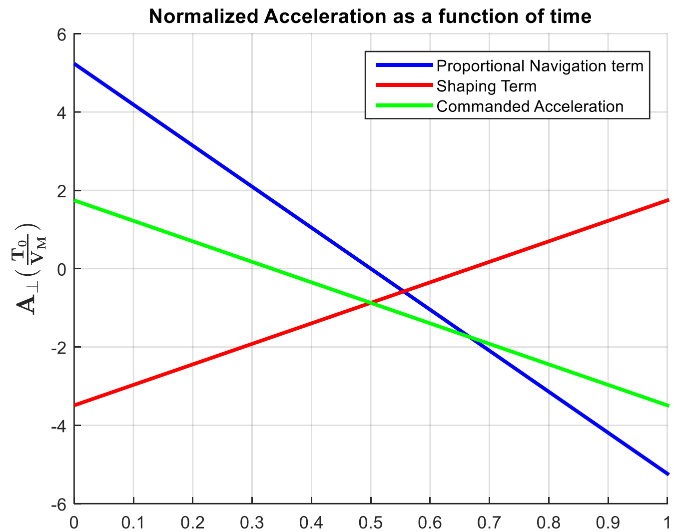
$$z = z_f + R\,(\delta - \gamma) \xrightarrow{Normalize\ by\ R_0} \bar{\mathrm{z}} = \frac{z}{R_0} \approx -\sigma_0 + \tau\,(\delta - \gamma)$$

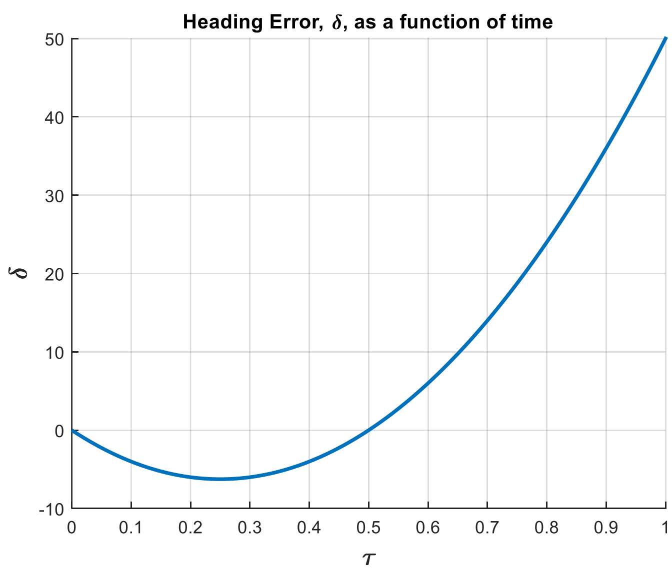
$$x = x_f - R \xrightarrow{Normalize\ by\ R_0} \bar{\mathrm{x}} = \frac{x}{R_0} \approx 1 - \tau$$

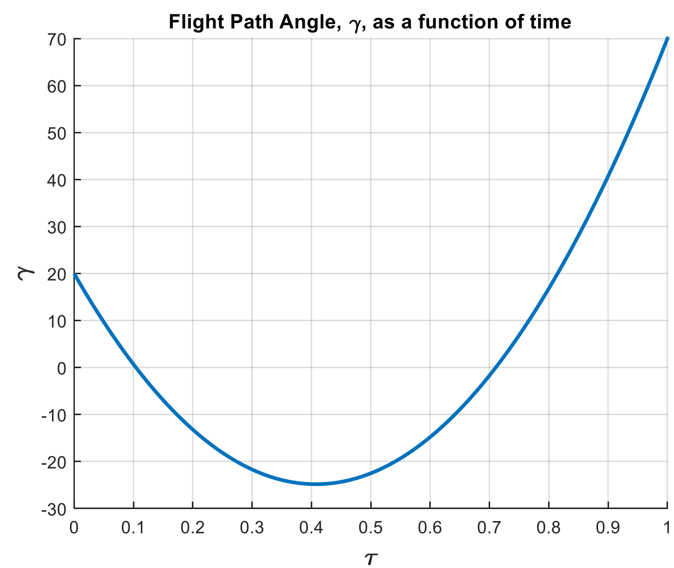
δ, γ, and $a_\perp$

OG to PIP with Shaping Constraint

- Due to the form of the control, it is not easy to develop a normalized function which describes the trajectory independent of scenario. Therefore, we evaluate the following scenario as an example:
  - Initial cond.: $\delta_0 = 50°,\ \gamma_0 = 70°$
  - Final cond.: $\delta_f = 0°,\quad \gamma_f = 20°$







Further Studies

OG to PIP with Shaping Constraint

- Just as we introduced a late maneuver penalty on the OG to a PIP problem, the same can be done with this problem

$$J = V_M^2 \int_0^{T_0} \frac{\left(\delta' + \frac{\delta}{T}\right)^2}{T^p}\, dT$$

- Recognizing the same constraints
  - Hit the target at $t = T_0$ (or $T = 0$)
  - Intercept the target with a prescribed flight path angle of γ_f
- Develop the optimal guidance law to a PIP with a shaping constraint that considers late maneuver penalty

General Form of δ, γ, and $a_\perp$

OG to PIP with Shaping Constraint

Shorthand Notation

- Guidance gains

$$\varphi = \begin{bmatrix} (p+2)(p+3) \\ (p+1)(p+2) \\ (p+1)(p+2)(p+3) \\ (p+1)(p+2)(p+2) \end{bmatrix}$$

- Error Terms

$$\varepsilon = \begin{bmatrix} \delta_0 \\ \gamma_f \quad - \gamma_0 \end{bmatrix}$$

- Remember, $K = p + 3$

The general form of the control ($A_\perp$), and state variables (δ, γ) for any maneuver penalty (p), over the region in which $p \succ -1$ is as follows:

$$\frac{A_\perp}{V_M/T_0} = (-\varphi_1\,\varepsilon_1 - \varphi_2\,\varepsilon_2)\tau^{p+1} + (\varphi_3\,\varepsilon_1 + \varphi_4\,\varepsilon_2)(1-\tau)\tau^p$$

$$\delta = \sum_{j=1}^{2} \varepsilon_j \left(\frac{\tau^{p+2}}{p+3}(\varphi_j + \varphi_{j+2}) - \frac{\tau^{p+1}}{p+2}\,(\varphi_{j+2}) \right)$$

$$\gamma = \gamma_0 + \sum_{j=1}^{2} \varepsilon_j \left(\frac{\varphi_{j+2}}{p+1}(1-\tau^{p+1}) - \frac{\varphi_j + \varphi_{j+2}}{p+2}\,(1-\tau^{p+2}) \right)$$

References

Lockheed Martin Material used as guide for this lecture (topics to cover), etc.

1. Corse, J.T. Midcourse Guidance Course. Lockheed Martin summer course, 1998

Further reading regarding optimization

1. Any good book on Calculus of Variations
2. Bryson and Ho. *Applied Optimal Control*. Taylor & Francis, 1975