Systems and Control

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Contents

Chapter 1

Control Systems: Introduction, Applications, Definitions

$$\text{Input} \longrightarrow \boxed{\text{System}} \longrightarrow \text{Output}$$

To a system to a desired state, compare the current state with the objective, pass the difference to a controller, and pass the output of the controller back to the system.

1.1 Basic Definitions & Lingo

Definition 1.1.1: System Modelling

A mathematical or input/output description of the behavior of a system.

Definition 1.1.2: Control

The use of information to affect the operation of a device, machine, or system of any kind.

Feedback is important to ensure that the objectives of the system are met.

Definition 1.1.3: Plant

The physical objective to control, impact or influence.

Definition 1.1.4: Control Objective

The desired behavior of a system.

Definition 1.1.5: Input

The signals used to control a plant.

Definition 1.1.6: Output

The measurements, data, and what is being sensed.

Definition 1.1.7: Process

The internal behavior of the plant as a result of the inputs.

Definition 1.1.8: Model

A mathematical description of the physics of the system.

Modelling is a useful tool for the engineering of a system. It can help to determine desired inputs, outputs, and processes.

Definition 1.1.9: Disturbances

Anything preventing the plant from achieving the desired output.

Every system will have external disturbances to deal with. The real world is never the same as the ideal world.

Example 1.1.1 (Traffic Control)

Plant: The transportation network—movement of cars, roads, connectivity, highways, physics of the network.
Processes: the movement of cars, switching of traffic lights
Control Objective: minimizing traffic
Input: The changing of the traffic light signals
Output: The movement of the cars
Disturbances: Accidents, snow, bad drivers

1.2 Control Strategies

Two common control strategies are the black box strategy and the model-based strategy.

Definition 1.2.1: Black Box Strategy

If the processes of a system are unknown, we can learn about the system by applying inputs and taking note of the outputs. While analysis is not possible on black box systems, there is no need for physical knowledge of the system.

Definition 1.2.2: Model-Based Strategy

Model-based strategies use mathematical models to describe the behavior of a system. Compared to the black box strategy, more knowledge of the system is required, but having this knowledge makes deeper analysis possible.

Using a model-based strategy, two control methods can be applied: the open loop method and the closed-loop method. The open loop method has no feedback mechanism, and is therefore susceptible to disturbances; however, open loop control systems are much simpler to build and model. On the other hand, closed loop systems have a feedback mechanism to reduce the impacts of disturbances. This mechanism adds complexity to the system, which makes modelling more difficult.

Chapter 2

Laplace Transforms, Transfer Functions, & ODEs

2.1 Laplace Transforms

The Laplace transform is a mathematical tool used to take a function of time, t, and transform it into a function of the complex frequency variable, s. The one-sided Laplace tranform is of the form:

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_{0}^{\infty} f(t)e^{-st}dt$$

Unfortunately, for some values of s, this integral is undefined.

Definition 2.1.1: Abscissa of Absolute Convergence

The *abscissa of absolute convergece* is the region in which there

Definition 2.1.2: Unit Impulse Function

The *unit impulse function*, $\delta(t)$ is given by:

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & \text{elsewhere} \end{cases}$$

The unit impulse function has the Laplace transform:

$$\mathcal{L}\left[\delta(t)\right] = \int_{0}^{\infty} \delta(t)e^{-st}dt = 1$$

Definition 2.1.3: Unit Step Function

The *unit step function*, $u(t)$ is defined as:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The unit step function has the Laplace transform:

$$\mathcal{L}\left[u(t)\right] = \frac{1}{s}$$

Definition 2.1.4: Unit Ramp Function

The *unit ramp function*, $v(t)$ is defined as:

$$v(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The unit ramp function has Laplace tranform:

$$\mathcal{L}[v(t)] = \frac{1}{s^2}$$

Example 2.1.1

1. Take the Laplace transform of $f(t) = 5 \forall t \geq 0$:

$$f(t) = 5u(t)$$

$$F(s) = \mathcal{L}[f(t)] = \frac{5}{s}$$

2. Take the Laplace transform of $f(t) = 2t \forall t \geq 0$:

$$f(t) = 5v(t)$$

$$F(s) = \mathcal{L}[f(t)] = \frac{2}{s^2}$$

3. Take the Laplace transform of $f(t) = e^{-at}$:

$$f(t) = e^{-at}u(t)$$

2.1.1 Linearity

By definition, Laplace transforms are a linear mapping.

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

Linearity of Laplace Transforms:

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = \int_0^\infty (a_1 f_1(t) + a_2 f_2(t))\, e^{-st} dt$$

By the distributive property of multiplication:

$$(a_1 f_1(t) + a_2 f_2(t))e^{-st} = a_1 f_1(t) e^{-st} + a_2 f_2(t) e^{-st}$$

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2.1.2 Differentiation

Differentiation is easy with a Laplace transform:

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

Where:

$f(0)$ is the initial condition of the function $f(t)$ at $t = 0$

In general:

$$\mathcal{L}[f^{(n)}] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

2.1.3 Final Value Theorem

Consider $F(s) = \frac{N(s)}{D(s)}$. The poles of $F(s)$ will occur at the roots of $D(s)$, and the zeros of $F(s)$ will occur at the roots of $N(s)$.

Theorem 2.1.1 Final Value Theorem

If all of the poles of $sF(s)$ occur only in the left half plane (LHP):

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Example 2.1.2

With all zero initial conditions for $y(t)$ and $u(t)$, a system is goverend by this second order ODE:

$$y''(t) + 3y'(t) + 2y(t) = 2u'(t) + u(t)$$

Using the final value theorem, find $\lim_{t \to \infty} y(t)$ if $u(t) = 1$

Take the Laplace transforms:

$$\mathcal{L}[y''(t)] = s^2 Y(s) - sy(0) - y'(0)$$
$$\mathcal{L}[y'(t)] = sY(s) - y(0)$$
$$\mathcal{L}[u'(t)] = sU(s) - u(0)$$

Plugging back in:

$$s^2 Y(s) + 3sY(s) + 2Y(s) = 2sU(s) + U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{2s + 1}{s^2 + 3s + 2}$$

Since $u(t)$ is the unit step, $U(s) = \frac{1}{s}$

$$Y(s) = \frac{1}{s} \cdot \frac{2s + 1}{s^2 + 3s + 2}$$

By the final value theorem of $y(t)$:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{2s + 1}{s^2 + 3s + 2} = \frac{1}{2}$$

2.1.4 Initial Value Theorem

Theorem 2.1.2 Initial Value Theorem

$$\lim t \longrightarrow 0^+ f(t) = \lim_{s \to \infty} sF(s)$$

If and only if the limit exists.

2.1.5 Convolution

$$f_1(t) * f_2(t) = \int$$

2.2 Partial Fraction Expansion

One way to compute the inverse Laplace transform is with partial fraction expansion. Given a transfer function, $Y(s)$, it can be broken down into the quotient of two simple transfer functions:

$$Y(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_0}$$

Example 2.2.1 (Partial Fraction Expansion: Distinct Roots)

$$F(s) = \frac{1}{s^2 + 5s + 6}$$

Factoring the denominator gives:

$$F(s) = \frac{1}{(s+3)(s+2)} = \frac{c_1}{s+3} + \frac{c_2}{s+2}$$

Find values for c_1 and c_2:

$$c_1 = \lim_{s \to -3} F(s)(s+3) = \lim_{s \to -3} \frac{1}{s+2} = -1$$

$$c_2 = \lim_{s \to -2} F(s)(s+2) = \lim_{s \to -2} \frac{1}{s+3} = 1$$

Plugging back in for $F(s)$ gives:

$$F(s) = \frac{-1}{s+3} + \frac{1}{s+2}$$

Example 2.2.2 (Partial Fraction Expansion: Repeated Roots)

$$F(s) = \frac{1}{(s+1)(s+2)^2}$$

Since the denominator is a third degree polynomial, there will be three partial fractions. Importantly, for repeated roots, there is a term for each power of the root:

$$F(s) = \frac{c_1}{s+1} + \frac{c_2}{s+2} + \frac{c_3}{(s+2)^2}$$

Find the values of the constants:

$$c_1 = \lim_{s \to -1} F(s)(s+1) = \lim_{s \to -1} \frac{1}{(s+2)^2} = 1$$

$$c_3 = \lim_{s \to -2} F(s)(s+2)^2 \lim_{s \to -2} \frac{1}{s+1} = -1$$

To solve for c_2, we must pick some random, preferably easy to work with, number to plug in for s:

$$F(0) = \frac{1}{4} = \frac{c_1}{1} + \frac{c_2}{2} + \frac{c_3}{3}$$

$$\frac{1}{4} = 1 + \frac{c_2}{2} + \frac{-1}{4}$$

$$c_2 = -1$$

To complete the inverse Laplace transform, plug in the constants:

$$F(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$f(t) = \left[e^{-t} - e^{-2t} + te^{-2t}\right] u(t)$$

Example 2.2.3 (Partial Fraction Expansion: Imaginary Roots)

$$F(s) = \frac{1}{s(s^2 + 2s + 2)}$$

Convert to partial fractions:

$$F(s) = \frac{c_1}{s} + \frac{As + B}{s^2 + 2s + 2}$$

Solve for constants:

$$c_1 = \lim_{s\to 0} F(s) = \frac{1}{2}$$

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{1}{2s} + \frac{As + B}{s^2 + 2s + 2}$$

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{(s^2 + 2s + 2) + 2s(As + B)}{2s(s^2 + 2s + 2)}$$

$$2 = (s^2 + 2s + 2) + 2As^2 + 2Bs$$

$$2 = (2A + 1)s^2 + 2s(B + 1) + 2$$

$$A = -\frac{1}{2}$$

$$B = -1$$

$$F(s) = \frac{1}{2s} + \frac{-\frac{1}{2}s - 1}{s^2 + 2s + 2}$$

2.3 Solving ODEs Using Laplace Transforms

Example 2.3.1

$$\ddot{y}(t) - y(t) = t$$

$$y(0) = \dot{y}(0) = 1$$

$$\mathcal{L}[\ddot{y}(t)] = s^2 y(s) - s\dot{y}(0) - y(0)$$

$$\mathcal{L}[y(t)] = y(s)$$

$$\mathcal{L}[t] = \frac{1}{s^2}$$

$$s^2 y(s) - s - 1 - y(s) = \frac{1}{s^2}$$

$$y(s)[s^2 - 1] = s + 1 + \frac{1}{s^2}$$

$$y(s)[s^2 - 1] = \frac{s^2(s + 1) + 1}{s^2}$$

$$y(s) = \frac{s^2(s + 1) + 1}{s^2(s^2 - 1)} = \frac{1}{s - 1} + \frac{1}{s^2(s^2 - 1)} = \frac{1}{s - 1} + \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s - 1} + \frac{c_4}{s + 1}$$

Chapter 3

Modeling of Dynamical Systems

The transfer function of a system is written in the form:

$$F(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + b_{n-1} s^{n-1} + \cdots + a_0}$$

The numerator of the system has order m, and the denominator has order n. We say that the order of the system is the same as the order of the denominator. The roots of the numerator are the zeros of the transfer function and are usually plotted with $\circ$. The roots of the denominator are the poles of the transfer function and are usually plotted as $\times$.

The goal of using this transfer function is to find the output, $y(t)$, of the system. Using the definition of the transfer function, this is done quite easily:

$$F(s) = \frac{Y(s)}{X(s)}$$

$$Y(s) = X(s)F(s)$$

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{X(s)F(s)\right\}$$

$$y(t) = x(t)f(t)$$

If $x(t)$ is the unit step, $u(t)$, the output is called the step response, if $x(t)$ is the Dirac delta function, $\delta(t)$, the output is called the impulse response, and if $x(t)$ is the ramp function, $v(t)$, the output is called the velocity response.

Chapter 4

Signal Flow Graphs

Signal flow graphs are an alternative to block diagrams. They may consist of nodes, paths, gains, and loops. A node is a place on the diagram with a value defined as the sum of its inputs. Nodes may exist as inputs to the system, outputs from the system, or simply a measurable point in the system. Paths connect nodes together, showing the direction that a signal travels. Gains describe how a signal transforms when travelling along a path from one node to another. Loops are a series of at least one path that start and end at the same node.

There exist some special types of paths and loops. A forward path connects the input directly to the output with no loops. A feedback path is just a loop, also called a feedback loop. A self-loop is a loop that connects a node back to itself without visiting any other nodes.

Mason's Rule:

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

where k is the number of forward paths, and T_k is the gain of the k^{th} forward-path, Δ is the determinant of the signal flow graph, and Δ_k is the associated path factor.