Homework 1 - Aidan Sharpe

1.

If $\vec{A} = 4\hat{x} + 4\hat{y} - 2\hat{z}$ and $\vec{B} = 3\hat{x} - 1.5\hat{y} + \hat{z}$, find the acute angle between $\vec{A}$ and $\vec{B}$.

Definition of dot product:

$$\vec{A} \cdot \vec{B} = (A_x B_x) + (A_y B_y) + (A_z B_z) = \|\vec{A}\| \|\vec{B}\| \cos(\varphi)$$

Solve the dot product:

$$\vec{A} \cdot \vec{B} = (4 \cdot 3) + (4 \cdot (-1.5)) + ((-2) \cdot 1) = 4$$

Magnitudes of the vectors:

$$\|\vec{A}\| = \sqrt{4^2 + 4^2 + (-2)^2} = \sqrt{36} = 6$$

$$\|\vec{B}\| = \sqrt{3^2 + \left(-\frac{3}{2}\right)^2 + 1^2} = \sqrt{\frac{49}{4}} = \frac{7}{2}$$

By the definition of the dot product:

$$4 = 6 \cdot \frac{7}{2} \cos(\varphi)$$

$$\therefore \cos(\varphi) = \frac{4}{21}$$

$$\therefore \varphi = \arccos\left(\frac{4}{21}\right) = 1.379 = 79.02^\circ$$

2

If $\vec{A} = \frac{10}{\rho}\hat{\rho} + 5\hat{\varphi} + 2\hat{z}$ and $\vec{B} = 5\hat{\rho} + \cos(\varphi)\hat{\varphi} + \rho\hat{z}$, find (a) $\vec{A} \cdot \vec{B}$ and (b) $\vec{A} \times \vec{B}$ at $x = 1$, $y = 1$, $z = 1$.

Convert from cartesian to cylidrical:

$$\rho = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\varphi = \arctan\left(\frac{y}{x}\right) = \frac{\pi}{4}$$

$$z = z = 1$$

Find $\vec{A}$ and $\vec{B}$ at $x = 1$, $y = 1$, $z = 1$:

$$\vec{A} = \frac{10}{\sqrt{2}}\hat{\rho} + 5\hat{\varphi} + 2\hat{z} = 5\sqrt{2}\hat{\rho} + 5\hat{\varphi} + 2\hat{z}$$

$$\vec{B} = 5\hat{\rho} + \cos\left(\frac{\pi}{4}\right)\hat{\varphi} + \sqrt{2}\hat{z} = 5\hat{\rho} + \frac{\sqrt{2}}{2}\hat{\varphi} + \sqrt{2}\hat{z}$$

a) Find $\vec{A} \cdot \vec{B}$

$$\vec{A} \cdot \vec{B} = \left(5\sqrt{2} \cdot 5\right) + \left(5 \cdot \frac{\sqrt{2}}{2}\right) + \left(2 \cdot \sqrt{2}\right) = \frac{59\sqrt{2}}{2}$$

b) Find $\vec{A} \times \vec{B}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\rho} & \hat{\varphi} & \hat{z} \\ 5\sqrt{2} & 5 & 2 \\ 5 & \frac{\sqrt{2}}{2} & \sqrt{2} \end{vmatrix} = 4\sqrt{2}\hat{\rho} - 20\hat{z}$$

3

Two point charges have mass 0.2g. Two insulating threads of length 1m are used to suspend the charges from a common point. The gravitational force is 980×10^{-5}N/g.

Define φ as the angle between the threads, α as half of that angle, and β as $\frac{\pi}{2} - \alpha$. This way, α and β make a right triangle with the leg adjacent to α making a perpendicular bisector of the distance between the charges, r.

Find the distance, r:

$$\frac{r}{2} = (1\text{m})\cos(\beta)$$
$$\therefore r = 2\cos(\beta)$$

By Coulomb's Law, the electric field force on each charge, $\vec{F}_e$, is:

$$\vec{F}_e = \frac{q^2}{4\pi\varepsilon_0 r^2}\hat{x}$$

Pluggin in for r,

$$\vec{F}_e = \frac{q^2}{16\pi\varepsilon_0 \cos^2(\beta)}\hat{x}$$

The force due to gravity on each charge, $\vec{F}_g$, is:

$$\vec{F}_g = 0.2 \cdot 980 \times 10^{-5} = 0.00196(-\hat{y})\text{N}$$

Assuming static equilibrium:

$$\sum F_x = 0 = F_{e_x} + F_{T_x} + F_{g_x}$$
$$\sum F_y = 0 = F_{e_y} + F_{T_y} + F_{g_y}$$

Where:

F_{T_x} is the x-component of the force due to tension in each thread, F_T.

F_{T_y} is the y-component of F_T.

Since F_g only acts in the $\hat{y}$ direction, and F_e only acts in the $\hat{x}$ direction:

$$F_{e_x} + F_{T_x} = 0$$

$$F_{T_y} + F_{g_y} = 0$$

Define the components of the tension force F_T:

$$F_{T_x} = F_T \cos\left(\frac{\pi}{2} + \alpha\right) = -F_T \cos(\beta)$$

$$F_{T_y} = F_T \sin\left(\frac{\pi}{2} + \alpha\right) = F_T \sin(\beta)$$

Solve for F_T using equilibrium in the $\hat{y}$ direction:

$$F_T \sin(\beta) - 0.00196 = 0$$

$$\therefore F_T = \frac{0.00196}{\sin(\beta)}$$

Plug in F_T to solve equilibrium in the $\hat{x}$ direction:

$$F_e + \left(\frac{0.00196}{\sin(\beta)}\right)(-\cos(\beta)) = 0$$

$$\therefore F_e = 0.00196 \cot(\beta)$$

a) When $\varphi = 45°$, solve for the charge, q:

Since $\varphi = \frac{\pi}{4}$, $\alpha = \frac{\pi}{8}$, and $\beta = \frac{3\pi}{8}$.

Known:

$$F_e = \frac{q^2}{16\pi\varepsilon_0 \cos^2(\beta)}$$

$$F_e = 0.00196 \frac{\cos(\beta)}{\sin(\beta)}$$

Set equal:

$$\frac{q^2}{16\pi\varepsilon_0 \cos^2(\beta)} = 0.00196 \cot(\beta)$$

$$\therefore q^2 = 16\pi\varepsilon_0 \frac{\cos^3(\beta)}{\sin(\beta)}$$

Plug in for β and solve for q:

$$q = \pm\sqrt{16\pi\varepsilon_0(0.00196)\frac{\cos^3\left(\frac{3\pi}{8}\right)}{\sin\left(\frac{3\pi}{8}\right)}} = \pm 2.300 \times 10^{-7}\text{C}$$

b) When $q = 0.5\mu$C, solve for the angle, φ:

Known:

$$F_e = \frac{q^2}{16\pi\varepsilon_0 \cos^2(\beta)}$$

$$F_e = 0.00196\frac{\cos(\beta)}{\sin(\beta)}$$

Set equal:

$$\frac{q^2}{16\pi\varepsilon_0 \cos^2(\beta)} = 0.00196\cot(\beta)$$

Plug in for q:

$$\frac{(0.5 \times 10^{-6})^2}{16\pi\varepsilon_0(0.00196)} = \frac{\cos^3(\beta)}{\sin(\beta)}$$

$$\therefore \frac{\cos^3(\beta)}{\sin(\beta)} = 0.287$$

$$\therefore \beta = 0.915$$

Using β to solve for φ:

$$\alpha = \frac{\pi}{2} - \beta = 0.656$$

$$\varphi = 2\alpha = 1.314 = 75.257°$$

4

The magnetic field, $\vec{B} = B_0(\hat{x} + 2\hat{y} - 4\hat{z})$ exists at a point. Find the electric field at that point if the force experienced by a test charge with velocity, $\vec{v} = v_0(3\hat{x} - \hat{y} + 2\hat{z})$ is 0.

Lorentz Force Law:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Since $\vec{F}$ is 0:

$$0 = q\vec{E} + q\vec{v} \times \vec{B}$$

$$\therefore q\vec{E} = -q\vec{v} \times \vec{B}$$

$$\therefore \vec{E} = -\vec{v} \times \vec{B}$$

By the definition of the cross product:

$$-\vec{v} \times \vec{B} = \vec{B} \times \vec{v}$$

In terms of $\vec{E}$:

$$\vec{E} = \vec{B} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & -4 \\ 3 & -1 & 2 \end{vmatrix} = -14\hat{y} - 7\hat{z}$$

5

A circular loop with radius, a, exists in the x-y plane. If the loop is uniformly charged and has total charge, Q, determine the $\vec{E}$-field intensity at some point along the axis normal to the loop.

By Coulomb's Law:

$$\vec{E} = \int \frac{dq}{4\pi\varepsilon_0 r^2}\hat{r}$$

Calling the distance along the normal axis z:

$$r^2 = a^2 + z^2$$

Along a uniformly charged line, the charge, dq, at any given point is given by the equation:

$$dq = \lambda dl$$

Where:

λ is the linear charge density

The linear charge density, λ is defined as:

$$\lambda = \frac{Q}{L}$$

Where:

Q is the total charge

L is the total length

For a circular loop with radius, a, and total charge, Q:

$$\lambda = \frac{Q}{2\pi a}$$

$$dl = ad\varphi$$

$$\therefore dq = \frac{Qd\varphi}{2\pi}$$

Plugging into Coulomb's Law:

$$\vec{E} = \int_0^{2\pi} \frac{Qd\varphi}{8\pi^2\varepsilon_0(a^2+z^2)}\hat{r}$$

$$\therefore \vec{E} = \frac{Q}{8\pi^2\varepsilon_0(a^2+z^2)}\int_0^{2\pi} \hat{r}d\varphi$$

Find $\hat{r}$ in terms of $\hat{x}$, $\hat{y}$, and $\hat{z}$:

$$\hat{r} = \frac{\vec{a} + \vec{z}}{\sqrt{a^2 + z^2}}$$

$$\vec{z} = z\hat{z}$$

$$\vec{a} = a\cos\varphi\hat{x} + a\sin\varphi\hat{y}$$

Back to Coulomb's Law:

$$\vec{E} = \frac{Q}{8\pi^2\varepsilon_0(a^2 + z^2)^{(3/2)}} \int\limits_0^{2\pi} (a\cos(\varphi)\hat{x} + a\sin(\varphi)\hat{y} + z\hat{z})d\varphi$$

By components:

$$E_x = \frac{aQ}{8\pi^2\varepsilon_0(a^2 + z^2)^{3/2}} \int\limits_0^{2\pi} \cos(\varphi)d\varphi = 0$$

$$E_y = \frac{aQ}{8\pi^2\varepsilon_0(a^2 + z^2)^{3/2}} \int\limits_0^{2\pi} \sin(\varphi)d\varphi = 0$$

$$E_z = \frac{zQ}{8\pi^2\varepsilon_0(a^2 + z^2)^{3/2}} \int\limits_0^{2\pi} d\varphi = \frac{zQ}{4\pi\varepsilon_0(a^2 + z^2)^{3/2}}$$

Recombining:

$$\vec{E} = \frac{zQ}{4\pi\varepsilon_0(a^2 + z^2)^{3/2}}\hat{z}$$

6

Consider a circular ring in the x-y plane with inner radius, a, outer radius, b, and uniform charge density, ρ_s. Find an expression for the $\vec{E}$-field at a point at distance, z, along the axis normal to the ring.

By Coulomb's Law:

$$d\vec{E} = \frac{dq}{4\pi\varepsilon_0 r^2}\hat{r}$$

Calling the radial distance, ρ:

$$r^2 = \rho^2 + z^2$$

For surface charge densities:

$$dq = \rho_s ds$$

In cylindrical coordinates:

$$ds = \rho d\rho d\varphi$$

Plugging into Coulomb's Law:

$$\vec{E} = \iint \frac{\rho_s \rho}{4\pi\varepsilon_0(\rho^2 + z^2)} \hat{r} d\rho d\varphi$$

$\hat{r}$ is defined as:

$$\hat{r} = \frac{\vec{\rho} + \vec{z}}{\sqrt{\rho^2 + z^2}}$$

Where:

$\vec{z} = z\hat{z}$

$\vec{\rho} = \rho\cos(\varphi)\hat{x} + \rho\sin(\varphi)\hat{y}$

Splitting $\hat{r}$ by components:

$$\hat{r} = \frac{\rho\cos(\varphi)\hat{x} + \rho\sin(\varphi)\hat{y} + z\hat{z}}{\sqrt{\rho^2 + z^2}}$$

Back to Coulomb's Law:

$$\vec{E} = \frac{\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi} \int_a^b \frac{\rho^2\cos(\varphi)\hat{x} + \rho^2\sin(\varphi)\hat{y} + \rho z\hat{z}}{(\rho^2 + z^2)^{3/2}} d\rho d\varphi$$

By components:

$$E_x = \frac{\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi} \int_a^b \frac{\rho^2\cos(\varphi)}{(\rho^2 + z^2)^{3/2}} d\rho d\varphi = 0$$

$$E_y = \frac{\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi} \int_a^b \frac{\rho^2\sin(\varphi)}{(\rho^2 + z^2)^{3/2}} d\rho d\varphi = 0$$

$$E_z = \frac{\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi} \int_a^b \frac{\rho z}{(\rho^2 + z^2)^{3/2}} d\rho d\varphi = \frac{\rho_s z}{2\varepsilon_0} \left(\frac{1}{\sqrt{z^2 + a^2}} - \frac{1}{\sqrt{z^2 + b^2}} \right)$$

Recombining:

$$\vec{E} = \frac{\rho_s z}{2\varepsilon_0} \left(\frac{1}{\sqrt{z^2 + a^2}} - \frac{1}{\sqrt{z^2 + b^2}} \right) \hat{z}$$

7

Consider two concentric cylindrical surfaces. The inner having radius, a, and charge density ρ_s, and the outer having radius, b, and charge density $-\rho_s$.

By Gauss's Law:

$$\Phi_E = \frac{Q_{enc}}{\varepsilon_0} = \oint_S \vec{E} \cdot d\vec{A}$$

The $\vec{E}$-field for a cylindrical surface with radius, ρ, and length, l, is given by the equation:

$$2\pi\rho l E = \frac{Q_{enc}}{\varepsilon_0}$$

a) For $\rho < a$:

$$Q_{enc} = 0$$
$$\therefore E = 0$$

b) For $a < \rho < b$:

$$Q_{enc} = 2\pi a l \rho_s$$

Where:

l is the length of the section of the cylender.

$$2\pi\rho l E = \frac{2\pi a l \rho_s}{\varepsilon_0}$$
$$\therefore E = \frac{a\rho_s}{\rho\varepsilon_0}$$

c) For $\rho > b$:

$$Q_{enc} = 2\pi l \rho_s (a - b)$$
$$2\pi\rho l E = \frac{2\pi l \rho_s (a - b)}{\varepsilon_0}$$
$$\therefore E = \frac{\rho_s (a - b)}{\rho\varepsilon_0}$$

8

Consider an infinite slab of thickness, d, centered on the origin ($x = 0$, $y = 0$, $z = 0$).

By Gauss's Law:

$$\Phi_E = \frac{Q_{enc}}{\varepsilon_0} = \oint_S \vec{E} \cdot d\vec{A}$$

a) Find the strength of the electric field inside the slab ($|z| < d/2$):

$$Q_{enc} = \rho_v lwh$$

$$\oint_S \vec{E} \cdot d\vec{A} = E(2lw + 2lh + 2wh)$$

$$E = \frac{\rho_v lwh}{2\varepsilon_0(lw + lh + wh)}$$

Where:

l is the size of the x dimension of a Gaussian rectangular prism centered on the origin,

w is the size of the y dimension of that rectangular prism,

h is the size of the z dimension of that rectangular prism

b) Find the strength of the electric field inside the slab ($|z| > d/2$):

$$Q_{enc} = \frac{\rho_v lwd}{2}$$

$$\oint_S \vec{E} \cdot d\vec{A} = E(2lw + 2lh + 2wh)$$

$$E = \frac{\rho_v lwd}{4\varepsilon_0(lw + lh + wh)}$$

Where:

l is the size of the x dimension of a Gaussian rectangular prism centered on the origin,

w is the size of the y dimension of that rectangular prism,

h is the size of the z dimension of that rectangular prism