Homework 5 - Aidan Sharpe

1

A nonuniform time-varying electric field given in the cylindrical coordinates by:

$$\vec{E} = \left(3\rho^2 \cot(\varphi)\hat{\rho} + \frac{\cos(\varphi)}{\rho}\hat{\varphi}\right)\sin(3\times 10^8 t)[\text{V/m}]$$

The field is applied to the following homogeneous, isotropic dielectric materials: 1. Teflon - $\mu_r = 1$ - $\varepsilon_r = 2.1$ - $\sigma = 0$ 1. Glass - $\mu_r = 1$ - $\varepsilon_r = 6.3$ - $\sigma = 0$ 1. Sea Water - $\mu_r = 1$ - $\varepsilon_r = 81$ - $\sigma = 4$[S/m]

a)

Find the polarization vector, the polarization current density, and the polarization charge density for each material.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$
$$\chi_e = \varepsilon_r - 1$$
$$\vec{J_p} = \frac{\partial}{\partial t}\vec{P}$$
$$\rho_p = -\nabla\cdot\vec{P}$$

In cylindrical coordinates:

$$\nabla = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$$

For teflon:

$$\vec{P} = \varepsilon_0(2.1-1)\left(9\rho^2\cot(\varphi)\hat{\rho} + \frac{\cos(\varphi)}{\rho^2}\hat{\varphi}\right)\sin(3\times 10^8 t)$$

$$\vec{J_p} = (3\times 10^8)\varepsilon_0(2.1-1)\left(9\rho^2\cot(\varphi)\hat{\rho} + \frac{\cos(\varphi)}{\rho^2}\hat{\varphi}\right)\cos(3\times 10^8 t)$$

$$\nabla\cdot\vec{P} = 1.1\varepsilon_0\sin(3\times 10^8 t)\left[\frac{\partial}{\partial\rho}9\rho^2\cot(\varphi) + \frac{\partial}{\partial\varphi}\frac{1}{\rho^2}\cos(\varphi)\right]$$

$$\rho_p = 1.1\varepsilon_0\sin(3\times 10^8)\left[-18\rho\cot(\varphi) + \frac{1}{\rho^2}\sin(\varphi)\right]$$

For glass:

$$\vec{P} = \varepsilon_0(6.3-1)\left(9\rho^2\cot(\varphi)\hat{\rho} + \frac{\cos(\varphi)}{\rho^2}\hat{\varphi}\right)\sin(3\times 10^8 t)$$

$$\vec{J}_p = (3\times 10^8)\varepsilon_0(6.3-1)\left(9\rho^2\cot(\varphi)\hat{\rho} + \frac{\cos(\varphi)}{\rho^2}\hat{\varphi}\right)\cos(3\times 10^8 t)$$

$$\nabla\cdot\vec{P} = 5.3\varepsilon_0\sin(9\times 10^8 t)\left[\frac{\partial}{\partial\rho}9\rho^2\cot(\varphi) + \frac{\partial}{\partial\varphi}\frac{1}{\rho^2}\cos(\varphi)\right]$$

$$\rho_p = 5.3\varepsilon_0\sin(9\times 10^8)\left[-18\rho\cot(\varphi) + \frac{1}{\rho^2}\sin(\varphi)\right]$$

For sea water:

$$\vec{P} = \varepsilon_0(81-1)\left(9\rho^2\cot(\varphi)\hat{\rho} + \frac{\cos(\varphi)}{\rho^2}\hat{\varphi}\right)\sin(3\times 10^8 t)$$

$$\vec{J}_p = (3\times 10^8)\varepsilon_0(81-1)\left(9\rho^2\cot(\varphi)\hat{\rho} + \frac{\cos(\varphi)}{\rho^2}\hat{\varphi}\right)\cos(3\times 10^8 t)$$

$$\nabla\cdot\vec{P} = 80\varepsilon_0\sin(3\times 10^8 t)\left[\frac{\partial}{\partial\rho}9\rho^2\cot(\varphi) + \frac{\partial}{\partial\varphi}\frac{1}{\rho^2}\cos(\varphi)\right]$$

$$\rho_p = 80\varepsilon_0\sin(3\times 10^8)\left[-18\rho\cot(\varphi) + \frac{1}{\rho^2}\sin(\varphi)\right]$$

b)

Find the ratio of the conduction to the displacement currents in the sea water.

$$\frac{4}{J_p} = \frac{1}{(6\times 10^9)\varepsilon_0\left(9\rho^2\cot(\varphi)\hat{\rho} + \frac{\cos(\varphi)}{\rho^2}\hat{\varphi}\right)\cos(3\times 10^8 t)}$$

NOTE: I am unsure why the book seems to be taking a slightly different approach to finding ρ_p. In the example we did in class, I was also off by a factor of three, so I am likely missing something here. The example we did in class was:

$$\vec{P} = k\vec{r} = kr\hat{r}$$

Taking the negative of the dot product with $\nabla = \frac{\partial}{\partial r}\hat{r}$ gives me $-k$ and not $-3k$, as you got. ## 2 Three coaxial cylinders separated by two different dielectric are charged as follows: - The inner cylinder of radius a has a positive linear charge density ρ_{l1}[C/m]. - The middle cylinder of radius b is grounded - The outer cylinder of radius c has negative linear charge density $-\rho_{l2}$[C/m]

a)

Determine and draw sketches showing the variation of electric flux density and the electric field intensity between the cylinders and outside them.

For the inner cylinder, the electric field inside is 0, and the flux through a cylindrical surface with length, z, and radius, ρ, enclosing the cylinder is given by Gauss's law:

$$\Phi_E = \frac{Q_{\text{enc}}}{\varepsilon_0 \varepsilon_r} = \frac{\rho_{l_1} z}{\varepsilon_0 \varepsilon_1} = \int \vec{E} \cdot d\vec{s}$$

By inspection, the electric field strength through the label of the cylinder is uniform, so the surface integral of the electric field evaluates to:

$$E 2\pi \rho z$$

Solving for the E-field strength at the surface yields

$$E = \frac{\rho_{l_1}}{2\pi\rho\varepsilon_0\varepsilon_1} : \{a < \rho < b\}$$

Since the middle cylinder is grounded the electric field is defined to be zero at and around its extent.

$$E = 0 : \{b \leq \rho \leq c\}$$

Outside the large cylinder, more charge is added, this time $-\rho_{l_2}$. The total electric field here is:

$$E = \frac{-\rho_{l_2}}{2\pi\rho\varepsilon_0} : \{\rho > c\}$$

The total electric field strength at a distance, ρ is given by the piecewise function:

$$E(\rho) = \begin{cases} \frac{\rho_{l1}}{2\pi\rho\varepsilon_0\varepsilon_1} & a < \rho < b \\ 0 & b \leq \rho < c \\ \frac{-\rho_{l_2}}{2\pi\rho\varepsilon_0} & \rho > c \end{cases}$$

Using some dummy values to plot the field strength. $\rho_{l1} = 1.5$, $\rho_{l2} = 1.3$, $\varepsilon_0 = 1$, $a = 1$, $b = 2$, $c = 3$, $\varepsilon_1 = 1.2$, $\varepsilon_2 = 1.4$.

$$E(\rho) = \begin{cases} \frac{\rho_{l1}}{2\pi\rho\varepsilon_0\varepsilon_1} & a < \rho < b \\ 0 & b \leq \rho < c \\ \frac{-\rho_{l_2}}{2\pi\rho\varepsilon_0} & \rho > c \end{cases}$$

b)

Determine the induced surface charge on the middle conductor.

$$\sigma(2\pi b z) = -\rho_{l_1} z$$

$$\therefore \sigma = -\frac{\rho_{l_1}}{2\pi b}$$

3

An N turn toroid of rectangular cross section consists of 3 regions. Region 1 with relative permeability $\mu_{r_1} = 3000$, region 2 with $\mu_{r_2} = 1 + 2/\rho$, region 3 is air.

a)

Find the magnetic field intensity, $\vec{H}$, the magnetic flux density, $\vec{B}$, and the magnetization, $\vec{M}$ in each of the three regions.

Region 1:

$$M_1 = (3000 - 1)\frac{NI}{2\pi\rho} = \frac{2998NI}{2\pi\rho}\hat{\varphi}$$

$$H_1 = \frac{N_I}{2\pi\rho}$$

$$B_1 = \mu_0\frac{3000NI}{2\pi\rho} = \frac{1500NI\mu_0}{\rho}\hat{\varphi}$$

Region 2:

$$M_2 = \frac{2}{\rho}\frac{NI}{2\pi\rho} = \frac{NI}{2\pi\rho^2}\hat{\varphi}$$

$$H_2 = \frac{N_I}{2\pi\rho}$$

$$B_2 = \mu_0\left(1 + \frac{2}{\rho}\right)\frac{NI}{2\pi\rho}\hat{\varphi}$$

Region 3:

$$M_3 = 0\hat{\varphi}$$

$$H_3 = \frac{N_I}{2\pi\rho}$$

$$B_3 = \mu_0\frac{NI}{2\pi\rho}\hat{\varphi}$$

b)

Find the magnetization current density in region 2:

$$J_m = \nabla \times \vec{H} = \frac{1}{\rho}\frac{\partial \vec{H}}{\partial \rho} = -\frac{NI}{\pi\rho^3}$$

4

A uniform plane wave is travelling in a medium in the $\hat{x}$ direction. $\lambda = 25$[cm], $v_p = 2 \times 10^8$[m/s], and $\hat{E}$ is polarized in the $\hat{z}$ direction.

a)

Find the frequency of the wave and ε_r:

$$\lambda = \frac{v_p}{f} \therefore f = \frac{v_p}{\lambda} = 8 \times 10^9 [\mathrm{s}^{-1}]$$

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}}$$
$$\mu = \mu_0$$
$$\varepsilon = \varepsilon_0 \varepsilon_r$$
$$\therefore \varepsilon_r = \frac{1}{\mu_0 \varepsilon_0 v_p^2} = 2.246$$

b)

$$\lambda = \frac{2\pi}{\beta} \therefore \beta = 8\pi$$
$$\hat{E} = 50 e^{j(16\pi \times 10^9 - 8\pi x)} \hat{z}$$
$$\hat{H} = -0.1989 e^{j(16\pi \times 10^9 - 8\pi x)} \hat{y}$$

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An electromagnetic plane wave of frequency f is given by:

$$\omega = 2\pi f$$
$$\hat{E} = E_0 e^{j\omega t - \beta z)} \hat{x}$$
$$\hat{H} = \frac{E_0}{\eta} \cos(\omega t - \beta z) \hat{y}$$

$$\mathcal{E} = -\frac{d}{dt} \mu \int_s \vec{H} \cdot d\vec{s}$$
$$\mathcal{E} = \mu \frac{E_0}{\eta} 2\pi\omega \sin(\omega t - \beta z) = \mu \frac{E_0}{\eta} 2\pi\omega e^{j(\omega t - \beta z - \frac{\pi}{2})}$$