Homework 2 - Aidan Sharpe

1

Consider a long cylindrical wire of radius, a, carrying a current $I = I_0 \cos(\omega t)\hat{z}$.

a)

Write an expression for the magnetic field strength, B, outside the wire ($\rho > a$):

By Ampere's Law:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

For a closed loop of radius, ρ:

$$\int \vec{B} \cdot d\vec{l} = B(2\pi\rho)$$

$$\therefore B(2\pi\rho) = \mu_0 I_{enc}$$

Since ($\rho > a$):

$$I_{enc} = I = I_0 \cos(\omega t)$$

Recombining:

$$B(2\pi\rho) = \mu_0 I_0 \cos(\omega t)$$

$$\therefore B = \frac{\mu_0 I_0 \cos(\omega t)}{2\pi\rho} [\text{T}]$$

b)

Consider a rectangular loop a distance, d, from the wire with sidelengths α in the $\hat{x}$ direction, and β in the $\hat{z}$ direction.

i) Calculate the magnetic flux, Φ_B, through the loop.

By Gauss's Law for magnetism:

$$\Phi_B = \iint \vec{B} \cdot d\vec{s}$$

Since the β does not vary:

$$\Phi_B = \beta \int_{d}^{d+\alpha} \frac{\mu_0 I_0 \cos(\omega t)}{2\pi\rho} d\rho$$

Taking out constants:

$$\Phi_B = \frac{\beta \mu_0 I_0 \cos(\omega t)}{2\pi} \int\limits_{d}^{d+\alpha} \frac{1}{\rho} d\rho$$

Evaluate:

$$\Phi_B = \frac{\beta \mu_0 I_0 \cos(\omega t)}{2\pi} \ln\left(\frac{|d+\alpha|}{|d|}\right) \text{ [Vs]}$$

ii) Find the induced EMF, $\mathcal{E}$:

$$\mathcal{E} = -\frac{d}{dt}\Phi_B$$

Plug in:

$$\mathcal{E} = -\frac{d}{dt}\frac{\beta \mu_0 I_0 \cos(\omega t)}{2\pi} \ln\left(\frac{|d+\alpha|}{|d|}\right)$$

Evaluate:

$$\mathcal{E} = \frac{\beta \mu_0 I_0 \sin(\omega t)}{2\pi} \ln\left(\frac{|d+\alpha|}{|d|}\right) \text{ [V]}$$

2

A quarter circle loop of wire with inner radius, a, and outer radius, b, has current, I. Find the magnetic field strength at the center of the circle.

By superposition:

$$B = B_1 + B_2 + B_3 + B_4$$

By Ampere's law:

$$\int \vec{B_1} \cdot d\vec{l} = \mu_0 I_{enc_1}$$

$$\int \vec{B_2} \cdot d\vec{l} = \mu_0 I_{enc_2}$$

$$\int \vec{B_3} \cdot d\vec{l} = \mu_0 I_{enc_3}$$

$$\int \vec{B_4} \cdot d\vec{l} = \mu_0 I_{enc_4}$$

Since the current in the first and third segments are either parallel or antiparallel:

$$I_{enc_1} = 0$$

$$I_{enc_3} = 0$$

Since all of I is enclosed in a loop from either segment two or four:

$$I_{enc_2} = I_{enc_4} = I$$

Back to Ampere's Law:

$$\int \vec{B_1} \cdot d\vec{l} = 0 \therefore B_1 = 0$$

$$\int \vec{B_2} \cdot d\vec{l} = \mu_0 I$$

$$\int \vec{B_3} \cdot d\vec{l} = 0 \therefore B_2 = 0$$

$$\int \vec{B_4} \cdot d\vec{l} = \mu_0 I$$

Determining $d\vec{l}$ for segments two and four:

$$\int \vec{B_2} \cdot \vec{a} d\varphi = \mu_0 I$$

$$\int \vec{B_4} \cdot \vec{b} d\varphi = \mu_0 I$$

Determining the bounds:

$$\int_0^{\frac{\pi}{2}} \vec{B_2} \cdot \vec{a} d\varphi = \mu_0 I$$

$$\int_{\frac{\pi}{2}}^{0} \vec{B_4} \cdot \vec{b} d\varphi = \mu_0 I$$

Evaluate:

$$B_2 \left(\frac{\pi a}{2} \right) = \mu_0 I$$

$$B_4 \left(\frac{-\pi b}{2} \right) = \mu_0 I$$

Solve for B:

$$B_2 = \frac{2\mu_0 I}{\pi a}$$

$$B_4 = -\frac{2\mu_0 I}{\pi b}$$

$$B = \frac{2\mu_0 I}{\pi a} - \frac{2\mu_0 I}{\pi b} = \frac{2\mu_0 I (a - b)}{\pi^2 ab} [\mathrm{T}]$$

3

Consider a cylinder of radius, $\rho_0 = 0.5$[m], with a current density, $\vec{J} = 4.5e^{-2\rho}\hat{z}[A/m^2]$.

By Ampere's Law:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

For a radial current density:

$$I_{enc} = \int_0^{2\pi} \int_0^{\rho} J s ds d\varphi$$

Plugging in J with s as the integrating variable:

$$I_{enc} = \int_0^{2\pi} \int_0^{\rho} 4.5e^{-2s} s ds d\varphi$$

$$I_{enc} = 4.5 \int_0^{2\pi} \int_0^{\rho} e^{-2s} s ds d\varphi$$

$$I_{enc} = 4.5 \int_0^{2\pi} \left[\left(\frac{-\rho}{2} - \frac{1}{4} \right) e^{-2\rho} + \frac{1}{4} \right] d\varphi$$

$$\therefore I_{enc} = \frac{9\pi e^{-2\rho}(e^{2\rho} - 2\rho - 1)}{4}$$

Back to Ampere's Law:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

For a circular loop:

$$\int_0^{2\pi} B \rho d\varphi = \mu_0 I_{enc}$$

$$B(2\pi\rho) = \frac{9\pi e^{-2\rho}(e^{2\rho} - 2\rho - 1)}{4}$$

For $\rho \leq \rho_0$

$$B = \frac{9e^{-2\rho}(e^{2\rho} - 2\rho - 1)}{8\rho}$$

For $\rho \geq \rho_0$:

$$I_{enc} = 4.5 \int_0^{2\pi} \int_0^{\rho_0} e^{-2s} s ds d\varphi = \frac{9(e-2)\pi}{4e}$$

$$B(2\pi\rho) = \frac{9(e-2)\pi}{4e}$$

$$\therefore B = \frac{9(e-2)}{8e\rho}[\text{T}]$$

4

Consider a solenoidal wire with n coils per unit length. The core becomes magnetized when a current $I = 10[A]$ is put into the wire coil, and this causes a bound current to flow around the cylindrical surface of the core as shown in the side view diagram. This bound core surface current density has magnitude, $J = 20n[A/m]$

$$B = B_s + B_c$$

$$B_s = \mu_0 In = 10\mu_0 n$$

Pretend the core current is just another solenoid:

$$B_c = \mu_0 In = 20\mu_0 n$$

$$B = 30\mu_0 n[\text{T}]$$

5

A satellite travelling at 5[km/s] enters a current filled curtain. From $t = 1$[s] to $t = 3$[s], the satellite's magnetometer increases from $-95\hat{x}$[nT] to $95\hat{x}$[nT]. If the current flows in the $\hat{z}$ direction, find the current density, J.

By Ampere's Law:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 \iint \vec{J} \cdot d\vec{s}$$

The total increase in B is 195[nT], so:

$$195 \times 10^{-9} = \mu_0 \iint \vec{J} \cdot d\vec{s}$$

Plug in bounds to get the total distance through the curtain:

$$195 \times 10^{-9} = \mu_0 \int_1^3 \int_0^{-5000} J dv dt$$

$$195 \times 10^{-9} = -10000 J \mu_0$$

$$J = \frac{-195 \times 10^{-13}}{\mu_0} [\mathrm{A/m}]$$