Discrete Fourier Transforms and Z-Transforms

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1 Results & Discussion

1.1 The Discrete Fourier Transform (DFT)

Given a signal, $x[n]$, it's N-point DFT is given by

$$X_k = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \tag{1}$$

where $W_N = e^{-j2\pi/N}$. The discrete Fourier transform is the sampled version of the discrete time Fourier transform (DTFT), which is a continuous function. More specifically, the N-point DFT contains N samples from the continuous DTFT.

For example, consider the signal $x[n] = (-1)^n$ for $0 \leq n \leq N-1$. By evaluating the sum shown in equation 1 as a truncated geometric series, the N-point DFT of $x[n]$ can be found. All truncated geometric series are evaluated as

$$\sum_{k=0}^{n-1} ar^k = \begin{cases} an & r = 1 \\ a\left(\frac{1-r^n}{1-r}\right) & r \neq 1 \end{cases}, \tag{2}$$

where r is the common ratio between adjacent terms. For the N-point DFT of $x[n]$, the common ratio is $-W_N^k$, which takes a value of 1 for $k = \frac{N}{2}$. Therefore, the N-point DFT of $x[n]$ is

$$X[k] = \begin{cases} N & k = \frac{N}{2} \\ \left(\frac{1-(-W_N^k)^N}{1-(-W_N^k)}\right) & k \neq \frac{N}{2} \end{cases}. \tag{3}$$

The N-point DFT of $x[n]$, where $N = 8$ is seen in figure 1. It only has a non-zero value for $k = \frac{N}{2} = 4$. This is the case for all even-number-point DFTs. Therefore, only odd-number-point DFTs should be used.

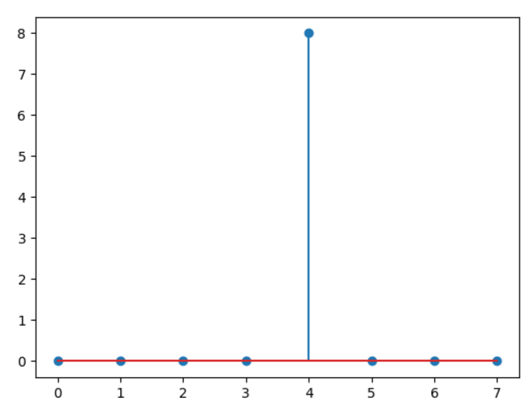

Figure 1: The N-point DFT of $x[n]$, where $N = 8$

For example, the 9-point DFT of $x[n]$, where $N = 8$ is seen in figure 2. While equation 3 cannot be used because there are a different number of samples for the DFT and the input signal, the overall DFT is more useful than the 8-point DFT.

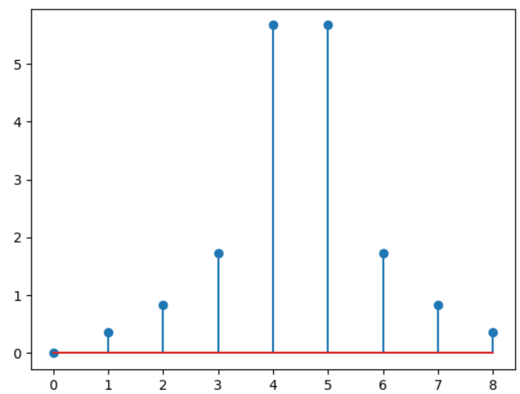

Figure 2: The 9-point DFT of $x[n]$, where $N = 8$

1.2 The Z-Transform

Given a discrete signal, $x[n]$, its z-transform is given by

$$X(z) = \sum_n x[n]z^{-n} \tag{4}$$

where z is a complex variable. The Z-transform of any discrete signal is a continuous function. Therefore, any attempt to calculate the Z-transform with a digital computer requires sampling or symbolic math. Symbolic math is quite a powerful tool, and others have worked hard to implement the Z-transform as a symbolic function. In MATLAB, this function is `ztrans`.

The Z-transform of the following signals can be found very quickly using this method:

$$x_1[n] = a^n u[n]$$

$$x_2[n] = (n+1)a^n u[n]$$

$$x_3[n] = a^n \cos(bn)u[n]$$

$$\mathcal{Z}\{x_1[n]\} = -\frac{z}{a-z}$$

$$\mathcal{Z}\{x_2[n]\} = \frac{az}{(a-z)^2} - \frac{z}{a-2}$$

$$\mathcal{Z}\{x_3[n]\} = -\frac{z(\cos(b) - z/a)}{a\frac{z^2}{a^2} - \frac{2z\cos(b)}{a} + 1}$$

1.3 The Inverse Z-Transform

Similarly, the inverse Z-transform also benefits from using a symbolic calculator. For example, the inverse Z-transform of the following can be found rapidly:

$$\mathcal{Z}\{x_4[n]\} = \frac{z}{z+0.5}$$

$$\mathcal{Z}\{x_5[n]\} = \frac{z^2}{(z-0.8)^2}$$

$$\mathcal{Z}\{x_6[n]\} = \frac{z}{(z+0.3)(z+0.6)^2}$$

$$x_4[n] = (-1/2)^n$$
$$x_5[n] = 2(4/5)^n(n-1)(4/5)^n$$
$$x_6[n] = \frac{50n(-3/5)^n - 100(-3/5)^n + 100(-3/10)^n}{9}$$

2 Conclusions

Overall understanding tricks for calculating geometric sums is very helpful for calculating DFTs. It is also important to use odd-number-point DFTs only, as taking an even number of samples forces almost all values to go to zero. As for Z-transforms and inverse Z-transforms, knowing how to use symbolic computation makes finding them quite easy. Additionally, symbolic computation allows for the easy manipulation of how simplified or expanded the result is.