Digital Signal Processing

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Contents

A signal is anything that is used to convey information. Analog signals are continuously variable, but in this course the focus will be put on digital signals. Digital signals are signals that are made up of discrete values. Some advantages of using digital signals are as follows:

1. Flexibility and programmibility
2. More immune to noise
3. Signal reproducibility
4. Ease of maintenance and troubleshooting
5. Signal storage

Furthermore, a signal can depend on any number of variables. A signal that only depends on one variable is considered *one-dimensional*. This course will primarily be directed towards these one-dimensional signals.

One example of the many use cases for digital signal processing is speech processing. This can include another range of topics such as speech recognition, speech enhancement, and speech encoding and compression.

Chapter 1

Discrete Time Signals

1.1 Uniform Sampling

The first step to uniform sampling is to discretize the time axis. Uniform sampling converts a continuous time signal, $x(t)$ into a discrete signal by considering the samples of $x(t)$ at uniform times, $t = nT_s$; where n is an integer and T_s is the sampling period.

The Dirac Delta function, or the impulse function is defined as:

$$\delta(t) = \begin{cases} 0 & t < 0 \\ \infty & t = 0 \\ 0 & t > 0 \end{cases}$$

The strength of the impulse function is typically defined to be unity. This is more rigorously given by:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

The sampling property of the impulse function is defined as:

$$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$$

The sifting property of the impulse function is given by:

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

To sample a signal, $x(t)$, uniformly through time, an impulse train is used. Each sample of the signal can be written in the form:

$$x(t)\delta(t - nT_s)$$

Therefore, the resulting sampled signal at all times, t, is given by:

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The signal, $x_{\text{imp}}(t)$, is a continuous and periodic function with period, T_s. It is given by:

$$x_{\text{imp}}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Since the definition of the Fourier series states that any continuous and periodic signal can be represented as a linear weighted combination of complex exponentials, we can rewrite $x_{\text{imp}}(t)$ in terms of a complex Fourier series:

$$x_{\text{imp}}(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_s t}$$

Where:

X_k are the Fourier series coefficients

Ω_s is the angular sampling frequency defined as $\Omega_s = \frac{2\pi}{T_s}$

Since Ω_s is entirely dependent on T_s, the only unknown is the values of the Fourier series coefficients, X_k. These coefficients can be found by:

$$X_k = \frac{1}{T_s} \int_{\frac{-T_s}{2}}^{\frac{T_s}{2}} \delta(t) e^{-jk\Omega_s t} dt$$

In this case, X_k is always $\frac{1}{T_s}$. Therefore, the Fourier series definition for $x_{\text{imp}}(t)$ is:

$$x_{\text{imp}}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\Omega_s t}$$

To sample $x(t)$ and get $x_s(t)$, simply take the product of $x(t)$ and $x_{\text{imp}}(t)$:

$$x_s(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} x(t) e^{jk\Omega_s t}$$

This sampled signal is called a discrete signal or a digital signal. While working with $x_s(t)$ in this form is not often practical, taking a Fourier transform gives more insight.

$$F[\delta(t-a)] = e^{-j\Omega a}$$

$$x(t) \overset{F}{\longleftrightarrow} X(\Omega)$$

$$x(t-\alpha) \overset{F}{\longleftrightarrow} X(\Omega)e^{-j\Omega\alpha}$$

$$x(t)e^{jn\Omega_0 t} \overset{F}{\longleftrightarrow}= X(\Omega - \Omega_0)$$

Using these properties, the Fourier transform of $x_s(t)$ is:

$$F[x_s(t)] = F\left[\frac{1}{T_s} \sum_{n=-\infty}^{\infty} x(t)e^{jn\Omega_s t}\right]$$

$$X_s(\Omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} F\left[x(t)e^{jn\Omega_s t}\right]$$

$$X_s(\Omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\Omega - n\Omega_s)$$

$$X_s(\Omega) = \frac{1}{T_s} X(\Omega)$$

Theorem 1.1.1 The Sampling Theorem

A band-limited signal, $x(t)$—its low-pass spectrum $X(\Omega)$ is such that $X(\Omega) = 0$ for $|\Omega| > \Omega_{\text{max}}$ where Ω_{max} is the maximum frequency in $x(t)$—can be sampled uniformly and without frequency aliasing (overlap between spectral copies) using a sampling frequency of $\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega$ called the Nyquist sampling rate condition.

Example 1.1.1 (Sampling Theorem)

A signal, $x(t)$, is given by:

$$x(t) = 2\cos(2\pi t + \frac{\pi}{4})$$

Since there exists a maximum frequency, in this case, 1[Hz], $x(t)$ is considered a band-limited signal.

$$f_s \geq 2f_m$$

$$f_s \geq 2[\text{Hz}]$$

$$T_s \leq \frac{1}{2}[\text{samples/s}]$$

A good figure of merit is to let $\Omega_{\max}$ be the frequency such that 99% of the signal energy is in the interval $[-\Omega_{\max}, \Omega_{\max}]$. The signal energy in the time domain is given by:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Using the Perseval's relationship, the signal energy can be computed using only knowledge from the frequency domain.

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 dt$$

1.2 Characterizing Discrete Signals in the Time Domain

Definition 1.2.1: Causal Signal

$x[n]$ is said to be a *causal signal* if $x[n]$ has zero value for $n < 0$.

When a signal, $x[n]$, is discritized following the sampling theorem, it domain is the integers, and it can take real and complex values.

Definition 1.2.2: Anti-Causal Signal

$x[n]$ is said to be an *anti-causal signal* if $x[n]$ has zero value for $n \geq 0$.

At $n = 0$, a causal signal can have a nonzero value, but an anti-causal signal cannot.

Definition 1.2.3: Finite Support Signal

$x[n]$ is said to have *finite support* if there exists integers, N_1 and N_2, such that $x[n]$ has zero value for $n < N_1$ and N_2, where $N_2 \leq N_1$.

Finite support signals have a finite domain, and for discrete signals, finite values of n that correspond to a non-zero signal value.

Definition 1.2.4: Infinite Support Signals

$x[n]$ is said to have *infinite support* or infinite duration if it does not have finite support.

For infinite support signals, there do not exist two integers, N_1 and N_2, such that $x[n] = 0$ for $n < N_1$ and $n > N_2$. Infinite support signals exist in the following forms:

1. Right-sided signal: $N < n < \infty$
2. Left-sided signal: $-\infty < n < N$
3. Two-sided signal: $-\infty < n < \infty$

Definition 1.2.5: Discrete Impulse

The impulse signal can be written in the discrete domain as:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since the discrete impulse signal has a non-zero value only at $n = 1$, it is a causal signal with finite support.

Example 1.2.1

Consider the signal, $x[n] = \delta[n - a]$:

$$\delta[n-a] = \begin{cases} 0 & n < a \\ 1 & n = a \\ 0 & n > a \end{cases}$$

Since $x[n]$ has a non-zero value only at $n = a$, it is a finite support signal. If $a \geq 0$, the signal is causal, but if $a < 0$, the signal is non-causal. In this case, if $a < 0$, then it is also an anti-causal signal.

Any finite support signal can be represented in terms of discrete impulses.

$$x[n] = \sum_{a=N_1}^{N_2} k_a \delta[n-a]$$

Definition 1.2.6: Discrete Unit Step

The *discrete unit step*, $u[n]$ is defined as:

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The discrete unit step has right-sided infinite support and is a causal signal. The signal, $u[-n]$ is defined as:

$$u[-n] = \begin{cases} 1 & n \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$u[-n]$ has left-sided infinite support, but it is neither causal nor anti-causal.

1.3 The Norm of a Discrete Signal

Definition 1.3.1: Norm of a Discrete Signal

The mapping of a signal to a value in the range, $[0, \infty)$.

Definition 1.3.2: The L_p Norm

Given a discrete time signal, $x[n]$, the L_p norm of the signal is:

$$\left[\sum_n |x[n]|^p\right]^{1/p}$$

Example 1.3.1 (L_p Norm)

$$x[n] = \begin{bmatrix} 3 & -5 & 7 & -5 & -9 \end{bmatrix}$$

Find the L_1, L_2, and L_∞ norms:

L_1 norm:

$$\sum_n |x[n]| = 3 + 5 + 7 + 5 + 9 = 29$$

L_2 norm:

$$\left[\sum_n |x[n]|^2\right]^{1/2} = sqrt9 + 25 + 49 + 25 + 81 = \sqrt{189} = 3\sqrt{21}$$

L_∞ norm:

$$\max |x[n]| = 9$$

Example 1.3.2 (L_p Norms with Complex Numbers)

$$x[n] = \begin{bmatrix} 3 + j & -5 + j3 & -7 - j & 9 - j4 & 10 \end{bmatrix}$$

L_1 norm:

$$\sum_n |x[n]| = \sqrt{10} + sqrt34 + \sqrt{50} + \sqrt{97} + 10$$

L_2 norm:

$$\left[\sum_n |x[n]|^2\right]^{1/2} = \sqrt{10 + 34 + 50 + 97 + 100}$$

L_∞ norm:

$$\max |x[n]| = 10$$

Example 1.3.3

$$x[n] = \begin{bmatrix} 1 & j & 1 & j & 1 & j & 1 & j & \cdots \end{bmatrix}, n \in \mathbb{Z} \cap [0, 99]$$

L_1 norm:

$$\sum_n |x[n]| = 100$$

L_2 norm:

$$\left[\sum_n |x[n]|^2\right]^{1/2} = \sqrt{100} = 10$$

L_∞ norm:

$$\max |x[n]| = 1$$

1.3.1 Geometric Series

There are two types of geometric series: finite and infinite.

Definition 1.3.3: Infinite Geometric Series

Infinite geometric series are of the form:

$$\sum_{n=m}^{\infty} s^n = \frac{s^m}{1-s}$$

If $|s| < 1$, the series will converge, and if $|s| > 1$, the series will diverge.

Definition 1.3.4: Finite Geometric Series

A finite geometric series is of the form:

$$\sum_{n=0}^{N-1} s^n$$

If $s = 1$, the series converges to N, otherwise the series can be evaluated by:

$$\sum_{n=0}^{N-1} s^n = \frac{1-s^N}{1-s}$$

Example 1.3.4 (Norms of Geometric Series)

$$x[n] = (-0.5)^n u[n]$$

L_1 norm:

$$\sum_n |x[n]| = \sum_{n=0}^{\infty} |(-0.5)^n| = \sum_{n=0}^{\infty} \frac{1}{2}^n = 2$$

L_2 norm:

$$\left[\sum_n |x[n]|^2\right]^{1/2} = \left[\sum_{n=0}^{\infty} |(-0.5)^n|^2\right]^{1/2} = \left[\sum_{n=0}^{\infty} \frac{1}{4}^n\right]^{1/2} = \left(\frac{1}{1-0.25}\right)^{1/2} = \sqrt{\frac{4}{3}}$$

L_∞ norm:

$$\max |x[n]| = \max(0.5^n), n \in \mathbb{N}_0 = 1$$

For geometric series, if $s < 1$, the L_∞ norm will occur at the smallest n in the series.Conversely, for geometric series where $s > 1$, the L_∞ norm will occur at the largest n in the series.

1.4 Elementary Operations on Signals

An elementary operation operates on each element of a signal. Consider two signals, $x[n]$ and $h[n]$. Assume that the two signals are sampled at the same frequency.

n	$x[n]$	$h[n]$	$x[n] + h[n]$	$x[n] - h[n]$	$x[n]h[n]$
-2	3		3	3	0
-1	-5	-7	-12	2	35
0	2	3	5	-1	6
1	-6	10	4	-16	-60
2	9	21	30	-12	289
3		-16	-16	16	0
4		6	6	-6	0
5		-3	-3	3	0

In addition to these element-by-element arithmetic operations, elementary operations may also manipulate the input space.

A signal may be time delayed by a in the form $h[n-a]$. Signals can be advanced by a in the form $h[n+a]$. Time may also be reflected around $n = 0$ in the form $h[-n]$. Perhaps the most interesting operation is the circular shift.

Definition 1.4.1: Circular Shift

The circular shift is a time shifting operation on a finite length sequence that results in another sequence of the same length and defined for the same range of values of n. The domain of the signal will be unchanged.

The modulus operation:

$$r = m \bmod N = \langle m \rangle_N$$

Another type of discrete elementary operation is the signal rate operation. This involves adding or removing samples.

Definition 1.4.2: The Downsample Operation

The downsample or sub-sample operation on a signal, $x[n]$ is given by:

$$y[n] = x[Mn], M \in \mathbb{Z}$$

This downsamples from every n to every M^{th} sample from $x[n]$

When downsampling, the sampling frequency f_s becomes $\frac{f_s}{M}$, and the sampling time, T_s becomes MT_s.

Note:

To avoid aliasing, $f_s \overset{!}{\geq} 2f_m$. Beware that downsampling may cause aliasing by decreasing f_s too much.

On the other hand, a signal may also be upsampled.

Definition 1.4.3: The Upsample Operation

Sampling frequency increases by a factor of M.

$$y[n] = x\left[\frac{n}{M}\right], M \in \mathbb{Z}$$

This will insert $M-1$ zeros between successive samples of $x[n]$.

Example 1.4.1 (Upsampling)

$$x[n] = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$
$$M = 4$$
$$y[n] = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & \cdots & 4 \end{bmatrix}$$

When upsampling, the sampling frequency, f_s becomes Mf_s, and the sampling time, T_s becomes $\frac{T_s}{M}$.

Chapter 2

Discrete Systems

A discrete system is any system that takes a discrete signal as input and produces a discrete signal as output.

$$x[n] \rightarrow \boxed{S} \rightarrow y[n]$$

2.1 Properties of Systems

2.1.1 Linearity

For a system to be considered linear, it must satisfy two conditions: homogeneity and superposition.

Homogeneity

For a system to be homogeneous, a scaling of the input by a constant, a, must produce an output scaled by that same constant, a.

$$y[n] = S[x[n]$$

$$ay[n] = S[ax[n]]$$

Superposition

For a system to satisfy superposition, taking the sum of two inputs $x_1[n]$ and $x_2[n]$ as input must produce the sum of their individual outputs $y_1[n] + y_2[n]$.

$$y_1[n] = S[x_1[n]]$$

$$y_2[n] = S[x_2[n]]$$

$$y_1[n] + y_2[n] = S[x_1[n] + x_2[n]]$$

2.1.2 Time Invarience

For a system to be considered time invariant, delaying or advancing the input must lead to the same delay or advance in the output.

$$y[n] = S[x[n]$$

$$y[n - a] = S[x[n - a]]$$

2.1.3 Causality

For a system to be considered causal, the output must only rely on past and present inputs; it cannot depend on any future inputs.

2.1.4 Bounded Input, Bounded Output (BIBO) Stability

A system is BIBO stable if every bounded input has a corresponding bounded output.

A bounded input is of the form:

$$|x[n]| \leq B_x < \infty, \forall n$$

A bounded output is of the form:

$$|y[n]| \leq B_y < \infty, \forall n$$

Note:
For unbounded inputs, outputs do not have to be bounded.

Example 2.1.1 (Properties of Downsampling)
The definition of the downsampling operation:

$$y[n] = S[x[n]] = x[Mn]$$

Homogeneity

$$S[ax[n]] = ax[Mn]$$

$$sy[n] = ax[Mn] \rightarrow \text{The system is homogeneous.}$$

Superposition

$$y_1[n] = S[x_1[n]] = x_1[Mn]$$

$$y_2[n] = S[x_2[n]] = x_2[Mn]$$

$$S[x_1[n] + x_2[n]] = x_1[Mn] + x_2[Mn] = y_1[n] + y_2[n] \rightarrow \text{Superposition holds.}$$

Time Invariance

$$S[x[n-a] = x[Mn-a]$$

$$y[n-a] = x[M(n-a)] = X[Mn - Ma] \neq S[x[n-a]] \rightarrow \text{The system is not time invariant}$$

Causality

For any value of M greater than 1, and n greater than 1, Mn will always be greater than n, therefore relying on future inputs.

BIBO Stability

If $|x[n]|$ is bounded by B_x for all n, then the output is bounded by $B_y = B_x$ for all n.

2.2 LTI Systems

Linear and time-invariant (LTI) systems are simple systems that are very useful for modelling.

Definition 2.2.1: Impulse Response

The output of a system when the Dirac delta (impulse) function is applied as input.

For discrete systems, the impulse response is given by $h[n]$ as opposed to the $h(t)$ for continuous systems.

Properties of the Discrete LTI Systems

1. The system is causal if the umpulse response is a causal signal

2. The system is BIBO stable if the impulse response is absolutely stable

$$\sum_n |h[n]| < \infty$$

3. $y[n]$ is computeable for any $x[n]$

The Integral Test

Given a continuous, positive, and decreasing function in the interval $[1, \infty)$, the indefinite integral and sum either both converge or both diverge.

Convolution

If the input to the LTI system is $x[n]$ and the impulse response of the LTI system is $h[n]$, the convolution of $x[n]$ and $h[n]$ is given by:

$$y[n] = x[n] * h[n]$$

In the continuous case, the convolution of $x(t)$ and $h(t)$ is given by:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

In the discrete case, the convolution of $x[n]$ and $h[n]$ is given by:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

2.3 Periodicity

A periodic continuous signal with period T is given by:

$$x(t) = x(t - kT)$$

A periodic discrete signal with integer period N is given by:

$$x[n] = x(t - kN)$$

The period of a discrete signal must be an integer because discrete signals are only defined at integer intervals.

If two discrete signals, $x[t]$ and $y[t]$ are periodic with periods N_1 and N_2, the period of the sum of the signals is the least common multiple of N_1 and N_2. If even one signal in a sum of signals is aperiodic, then the entire sum becomes aperiodic.

In the continuous time domain, sinusoidal and complex exponential signals are always periodic. For discrete signals, however, it is not as simple. A cosine in the discrete domain takes the form:

$$x[n] = \cos(\omega_0 n + \varphi)$$

By the definition of periodicity, for $x[n]$ to be periodic with period N:

$$x[n] = x[n + N] = \cos(\omega_0(n + N) + \varphi)$$

By distributing ω_0 and applying a trigonometric identity:

$$x[n + N] = \cos(\omega_0 n + \varphi)\cos(\omega_0 N) - \sin(\omega_0 n + \varphi)\cos(\omega_0 N)$$

Therefore, for $x[n]$ to equal $x[n+N]$, the sine terms must go to 0 and $\cos(\omega_0 N)$ must be unity. If ω_0 is an integer multiple of 2π, this condition will be satisfied.

Example 2.3.1 (Periodic Sampling)
Given a signal $\cos(\pi n)$, what is it's corresponding integer period?

$$\omega_0 = \pi$$

$$\omega_0 N = 2\pi r$$
$$\pi N = 2\pi r$$
$$N = 2r$$

The smallest integer r resulting in an integer N is $r = 1$. For $r = 1$, $N = 2$, therefore the signal is periodic with $N = 2$.

Example 2.3.2 (Aperiodic Sampling)
Given the signal $\cos(e\pi n + 7\pi/9)$ find it's corresponding integer period.

$$\omega_0 = e\pi$$

$$\omega_0 N = 2\pi r$$
$$e\pi N = 2\pi r$$
$$N = \frac{2}{e} r$$

There is no integer r such that N is an integer. Therefore the original signal is not discretely periodic.

2.4 Energy and Power

2.4.1 Energy

For a continuous signal, $x(t)$, the total energy is given by:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Similarly, for a discrete time signal, the total energy is given by:

$$E_x = \sum_n |x[n]|^2$$

which is the same as the square of the L-2 norm of $x[n]$.

2.4.2 Power

The power of a signal is calculated slightly differently for periodic and aperiodic signals. For discrete, periodic signals with period, N, the power is given by:

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

For aperiodic discrete signals, the power is given by:

$$P_x = \lim_{k \to \infty} \frac{1}{2k+1} \sum_{n=-k}^{k} |x[n]|^2$$

Note:

Periodic signals have infinite energy and finite power, while aperiodic signals with finite energy have zero power.

Example 2.4.1

Given $[x] = 3(-1)^n u[n]$ compute the energy and power. Since $x[n] = 0$ for $n < 0$, the signal is aperiodic.

$$E_x = \sum_n |x[n]|^2$$

$$E_x = \sum_{n=0}^{\infty} |3(-1)^n|^2$$

$$E_x = \sum_{n=0}^{\infty} 9 \to \infty$$

Given that the signal is aperiodic, the power is given by:

$$P_x = \lim_{k\to\infty} \frac{1}{2k+1} \sum_{n=-k}^{k} |x[n]|^2$$

$$P_x = \lim_{k\to\infty} \frac{1}{2k+1} \sum_{n=0}^{k} 9$$

$$P_x = \lim_{k\to\infty} \frac{1}{2k+1} 9(k+1)$$

$$P_x = \lim_{k\to\infty} \frac{9(k+1)}{2k+1} 9(k+1)$$

By L'hopital's rule,

$$P_x = \lim_{k\to\infty} \frac{9}{2} = \frac{9}{2}$$

Chapter 3

3.1 The Discrete Time Fourier Transform

3.1.1 Derivation of the Discrete Time Fourier Transform

Consider the signal $x_s(t)$ given by:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

Apply a Fourier transform:

$$F[x_s(t)] = F\left[\sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)\right]$$

By the sampling property of the impulse, $x(t)\delta(t - nT_s)$ becomes $x(nT_s)\delta(t - nT_s)$.

$$F[x_s(t)] = F\left[\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)\right]$$

The sum and $x(t)$ may be taken out of the Fourier transform:

$$F[x_s(t)] = \sum_{n=-\infty}^{\infty} x(nT_s)F\left[\delta(t - nT_s)\right]$$

The Fourier transform of the impulse is unity, and the delay by nT_s becomes a complex exponential:

$$X_s(\Omega) = \sum_{n=-\infty}^{\infty} x[nT_s]e^{-j\Omega nT_s}$$

If T_s is unity, then simplification is possible. Converting $x[nT_s]$ to $x[n]$ yields the discrete time Fourier transform:

$$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Note:

Ω is the continuous angular frequency while ω is the discrete angular frequency.

$$\Omega T_s = \omega$$

This is a natural extension of the continuous Fourier transform:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-\Omega t}dt$$

$X(e^{j\omega}$ is periodic with a period of 2π.

Definition 3.1.1: Discrete Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Example 3.1.1 (DTFT of the Impulse)
Consider $x[n] = \delta[n]$. Find the DTFT of $x[n]$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n}$$

The expression in the sum has a non-zero value only at $n = 0$, therefore the sum only has one term.

$$X(e^{j\omega}) = e^{-j\omega(0)} = 1$$

Example 3.1.2 (More Complicated DTFT)
Consider $x[n] = a^n u[n]$. Find the DTFT of $x[n]$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Since $u[n]$ only has a non-zero value for $n \geq 0$, the sum can be simplified to:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

Since both terms in the sum have n in the exponent:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

The sum has the form of a geometric series, which can be evaluated by:

$$\sum_{n=m}^{\infty} = \frac{s^m}{1-s}, |s| < 1$$

Since $|e^{-j\omega}| = 1$, $|s| = |a|$. Therefore:

$$F\left[a^n u[n]\right] = \frac{1}{1-ae^{-j\omega}}, |a| < 1$$

3.1.2 Properties of the Discrete Time Fourier Transform

1. The DTFT is a linear operation.
2. If a signal is delayed or advanced ($x[n-a]$), the DTFT is scaled by $e^{-j\omega a}$.
3. The DTFT of $nx[n]$ is $j\frac{d}{d\omega}(X(e^{j\omega}))$
4. The DTFT of the convolution, $x[n] * h[n]$, is the same as the product of the DTFT of the two signals.

Example 3.1.3 (Time Multiplication Property)

Consider $x[n] = na^n u[n]$. Find the DTFT of $x[n]$. Since the DTFT of $a^n u[n]$ is known to be $\frac{1}{1-ae^{-j\omega}}$, by the time multiplication property of the DTFT, the DTFT of $x[n]$ is:

$$X(e^{j\omega}) = j\frac{d}{d\omega}\left[\frac{1}{1-ae^{-j\omega}}\right]$$

Note:

If $x[n]$ is absolutely summable ($\sum |x[n]| < \infty$) then the DTFT exists.

3.1.3 Special Cases

In some special cases, the DTFT exists, but the corresponding time domain signal is not absolutely summable.

Example 3.1.4 (Ideal Lowpass Filter)

Consider the transfer function of an ideal lowpass filter, with cutoff frequency, ω_c.

Chapter 4

The Z-Transform

Recall the Laplace transform is defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

where s is a complex variable. The analogous transform in the discrete domain is called the Z-transform, which is given by:

$$X(z) = \sum_{n} x[n]z^{-n}$$

where z is a complex variable. The region of convergence of a Z-transform is the set of all z for which the sum converges.

Example 4.0.1 (Z-Transform of causal finite support signals)

Given a causal, finite support signal, $x[n] = \begin{bmatrix} a & b & c & d \end{bmatrix}, 0 \le n \le 3$. Find its z transform.

$$X(z) = \sum_{n=0}^{3} x[n]z^{-n}$$

$$X(z) = a + bz^{-1} + cz^{-2} + dz^{-3}$$

The region of convergence is all values of z except $z = 0$.

Example 4.0.2 (Z-Transform of anti-causal finite support signals)

Given the finite support, anti-causal signal, $x[n] = \begin{bmatrix} a & b & c & d \end{bmatrix}, -4 \le n \le -1$.

$$X(z) = \sum_{n=-4}^{-1} x[n]z^{-n}$$

$$X(z) = az^{4} + bz^{4} + cz^{2} + dz$$

The region of convergence is all values of z except $z = \infty$.

Example 4.0.3 (Z-Transform of causal nor anti-causal signals)

Given the finite support signal, $x[n] = \begin{bmatrix} a & b & c & d \end{bmatrix}, -2 \le n \le 1$.

$$X(z) = \sum_{n=-2}^{1} x[n]z^{-n}$$

$$X(z) = az^{2} + bz + c + dz^{-1}$$

The region of convergence is all z except $z = 0$ and $z = \infty$

For finite support, discrete signals, the region of convergence is always the entire z-plane with the possible exception of $z = 0$ and $z = \infty$.

Example 4.0.4 (Z-Transform of the discrete unit impulse)
Given $x[n] = \delta[n]$:

$$X(z) = \sum_{n} = \delta[n]z^{-n}$$

$$X(z) = 1$$

The region of convergence is the entire z plane.

Example 4.0.5 (Z-Transform of infinite support signals)
Given the infinite support signal, $x[n] = a^n u[n]$:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

The common exponent, n can be taken out to put it in the form:

$$\sum_{n=m}^{\infty} s^n = \frac{s^m}{1-s}$$

where $s = az^{-1}$ in this case. This sum will converge as long as $|az^{-1}| < 1$. In simpler terms, the sum will converge when $|z| > |a|$.

Since z is a complex variable, it can be described in a polar form:

$$z = re^{j\theta}$$

where r is the magnitude and θ is the phase angle.

If a region of convergence does not include the unit circle, then its DTFT does not exist except for special case signals. If the ROC *does* contain the unit circle, then the DTFT can be found by substituting $z = e^{j\omega}$.