Kappa Guidance

(and an Introduction to the Pontryagin Maximum Principle)

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Pontryagin's Principle

- Before diving headlong into the derivation of Kappa guidance, it is prudent to discuss the Pontryagin Principle
- This means of optimization was developed by Lev Pontryagin and his students
  - Developed in Russia in the mid 1950s, with the name "Pontryagin's Maximum Principle"
  - This optimization method was not well known outside of scholarly circles in the USA for decades after its development due to the ongoing cold war between the Russia and the USA
  - The Euler-Lagrange equation is a special case of the Pontryagin Principle
- Core principles
  - Uses the Hamiltonian and Lagrangian of the system, in conjunction with costates to determine an optimal control (can be minimized or maximized)
  - There are as many as 5 conditions for the Pontryagin Principle to optimize a system
    - All 5 conditions may not be necessary
    - We'll discuss the conditions and then use the Pontryagin Principle to develop Kappa guidance

Pontryagin Parameters

- The Pontryagin Principle can be used for problems far more complicated than the one we're about to tackle. We wont use all these parameters, but they are provided in the table for completeness

Symbol	Parameter	
t	Independent variable	✓
x	State vector	✓
u	Control variable	✓
e(x$_f$, t$_f$)	Endpoint equality constraint vector	✓
E(x$_f$, t$_f$)	Endpoint cost (Mayer cost) vector	✗
N(x, t)	Interior equality constraint vector	✗
C(x, u, t$_f$)	Control variable equality constraint	✗

The ✓ and ✗ symbols indicate the parameters needed to derive the original kappa guidance law

- The general case also requires a number of Lagrangian multipliers

Symbol	Parameter	
	Endpoint constraint covector	✓
	Control constraint covector	✗
	Interior constraint covector	✗

Conditions of the Pontryagin Principle

- Define the Lagrangian (L), full cost function (J), and Hamiltonian (H)

$L =$ the integrand of the traditional cost function (the running cost)

$$J_{full} = E(x_f, t_f) + \boldsymbol{\nu}^T \boldsymbol{e}(x_f, t_f) + \boldsymbol{\Pi}^T \boldsymbol{N}(x,t) + \int L(x,u,t)\,dt$$

$$H = L(x,u,t) + \boldsymbol{\lambda}^T f(x,u,t) + \boldsymbol{\mu}^T \boldsymbol{C}(x,u,t)$$

- The optimal control is determined by setting the partial derivative of the Hamiltonian with respect to the control to zero :

Hamiltonian Minimization Condition: $\frac{\partial H}{\partial u} = 0$

- However, to solve for the optimal control, it is often necessary to use costates (Pontryagin referred to them as covectors)
  - The costates exist in a different vector space which makes the optimization problem easier
  - Covectors (or costates) are often referred to as adjoints

Conditions of the Pontryagin Principle

- The introduction of the costates means a method to traverse between "x" state vector space and "λ" state (costate) vector space. The relationship between these two vector spaces are described in the following set of equations

Terminal Transversality Condition: $\boldsymbol{\lambda}^T(t_f) = \frac{\partial E(x,t)}{\partial \boldsymbol{x}} + \nu^T \frac{\partial\, e(x,t)}{\partial \boldsymbol{x}}$

Adjoint Equations: $\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \boldsymbol{x}}$

- In the event that the endpoints of the independent variable, t, is not fixed, an additional equation is required to complete the set of equations to solve for optimality. This equation is the Hamiltonian Value Condition:

Hamiltonian Value Condition: $H(t_f) = -\left(\frac{\partial E(x,t)}{\partial\, t_f} + \nu^T \frac{\partial\, e(x,t)}{\partial\, t_f}\right)$

- Finally, to validate the solution, the Hamiltonian Evolution Equation can be used

Hamiltonian Evolution Equation: $\frac{d}{dt}[\mathcal{H}] = \frac{\partial H}{\partial t}$

Where $\mathcal{H}$ is Hamiltonian with the optimal value of u in the equation

Using the Pontryagin Principle

- Some care must be used when using the Pontryagin Principle
  - Improper problem set-up is often an issue
  - Proper characterization of the problem is key
- Now that some background has been given, the kappa guidance law will be derived using the Pontryagin Principle
- Further reading on the subject of optimization and the Pontryagin Principle is readily available. Two suggested books are
  - Ross, I. Michael. *A Primer on Pontryagin's Principle in Optimal Control*. 2009
  - Bryson and Ho. *Applied Optimal Control*. Taylor & Francis, 1975

The Flaw in n_z^2 Minimization as a Cost

- The majority of optimal control guidance laws in use for endo-atmospheric interceptors are based upon minimizing the induced drag of the interceptor

$$\text{minimize} \int_0^{T_0} D_i \; dt \;\propto\; \int_0^{T_0} n_z^2 \; dt$$

- As mentioned in earlier lectures, this assumes the zero-lift drag is insignificant compared to the induced drag
  - This is only a reasonable assumption for short range engagements
- Furthermore, the usage of time as an independent variable is not as natural as a parameter which describes the interceptor's path in which drag accrues
  - The most natural independent variable is the interceptor's trajectory path, s
  - Range to go (R) is a suitable alternate which is much easier with which to work than time, t, or time-to-go, T

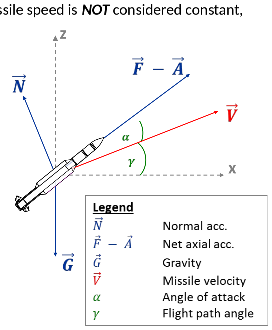
Traditional Assumptions Used in Guidance Law Optimization Theory Become Poorer as the Path Length of the Trajectory Increases

Optimization Hurdles

- Using the path of the trajectory, s, as the independent variable is something not discussed previously in our lectures, but critical if one is to consider the affect of zero-lift drag in the control law
- Our two contributors to the loss of missile speed over the entire trajectory, in terms of s, can be written as such:
  - Zero-Lift Drag, $D_{ZLD} \cong \frac{1}{2} \int_{S_0}^{0} C_A \, \rho \, V^2 \, S_{Ref} \; ds$
  - Induced drag, $D_i \cong \frac{1}{2} \int_{S_0}^{0} \frac{n_z^2 \, W^2}{(C_{N_\alpha}) \, \rho \, V^2 \, S_{Ref}} \; ds$
- Attempting to develop a cost function which is dependent upon highly non-linear parameters such as ambient density, ρ, and interceptor speed, V, posed a significant hurdle when developing an optimal control law

Back to Basics

- In order to properly set up this problem given that missile speed is ***NOT*** considered constant, one goes back to basics
- Consider the 2 dimensional diagram of the physical orientation of a missile and its accelerations
  - Acceleration are in blue
  - Angles are in green
  - Other parameters of interest are in red
  - Values of vectors without arrows are the magnitude of the vector (e.g. $N = \|\vec{N}\|$
- The normal force is the control that turns the missile. It's corresponding acceleration, N has been defined previously
  - It is a function of missile speed
  - $N = V_M \, \dot{\gamma}$



Removal of the Speed Dependency

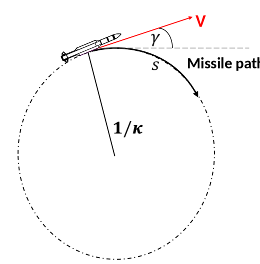
- Having defined s to be the trajectory arc-length, we can define the instantaneous curvature as κ and the instantaneous radius of curvature as $1/\kappa$
- From principles of circular motion,

$$\frac{d\gamma}{dt} = V \frac{d\gamma}{ds}$$

$$\kappa = \frac{d\gamma}{ds} = \frac{1}{V}\frac{d\gamma}{dt}$$

$$N = V^2 \, \kappa$$

- If one considers κ to be the control rather than N, one can remove the dependency upon missile speed in the guidance optimization process



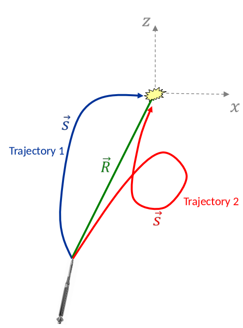
The Concept of Optimizing Curvature Vice Acceleration Commands is Why the Guidance Law is Called Kappa Guidance

Removal of the Time Dependency

- The time it takes to reach an intercept point is determined by the path taken by the missile to the intercept point and the speed of the missile as it moves through the atmosphere

$$T_0 = \int_{S_0}^{0} V \; ds$$

  - An unknown optimal trajectory or a poor estimate of missile speed over the trajectory obviously has an unknown time of flight
- Considering range to go (R) to be the independent variable removes dependency upon "when" the missile will achieve intercept
  - It is a reasonable decision when the trajectory is not circuitous (trajectory 2)
  - Also, R is an easier parameter to work with than s, as its value is always known



Summary of State Variables

- To develop a guidance law which considers zero-lift drag, one could define the state variables and independent variable as is done traditionally, or define the problem more efficiently, as has been discussed in this lecture
- The efficient method not only removes the dependencies of time and speed, but it also reduces the number of state variables required. It is the reduction of state variables which allows one to say that this method is more "efficient"
- The auxiliary state variable γ is needed as kappa guidance considers a prescribed final flight path angle, γ_f

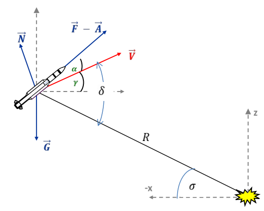
Parameter	Traditional Method	Efficient Method
Independent Variable	t	R
Control	N	κ
State Variables	V, x, z, δ	δ
Auxiliary State Variables	γ	γ

Kappa Guidance

- Let's return back to the problem at hand – Kappa guidance
- Kappa guidance is an attempt to consider the zero-lift drag on a projectile as it is guided along a path to a specified intercept point
- Kappa guidance was developed at RCA in Moorestown, NJ in the mid 1970s
  - It has since been implemented in numerous tactical missile systems
  - It has been analyzed, discussed and evaluated in open literature since its initial publication in open literature (Chin-Fang Lin, Modern Navigation Guidance and Control Processing, Prentice-Hall, 1991)
- Kappa guidance is most simply (and was originally) derived using the Pontryagin Principle

Defining the Geometry

- The figure to the right shows the missile's relationship to its intended intercept point as well as angular definitions and accelerations acting upon the missile
- The necessary geometric definitions are provided



Cartesian Definitions

$R = \sqrt{x^2 + z^2}$

$x = -R\cos\sigma$

$z = \quad R\sin\sigma$

$\frac{dx}{ds} = \cos\gamma$

$\frac{dz}{ds} = \sin\gamma$

Angle Definitions

$\delta = \gamma + \sigma$

$\sigma = -\tan^{-1}\frac{z}{x}$

$\delta = \gamma \; - \tan^{-1}\frac{z}{x}$

Control Relationship

$\frac{d\gamma}{ds} = \kappa$

State Equations of Motion

- Having defined the relationships between the Cartesian parameters (x, y) and the angular parameters (δ, γ, and σ), one must define the state variables in terms of the independent variable, R
- Because the majority of the geometric definitions are a function of path length, s, rather than our independent variable, R, Eq. 1 is required to describe the equations of motion in terms of the range-to-go, R
- Note that the state equations (Eq. 3 and Eq. 4) do not depend upon missile speed (V) or time (t or T).
- Also note that the equations are not linear. That may be a problem later…

- From the definitions on the previous slide, one can derive the following relationship between the path length and the independent variable, R

Eq. 1 $\quad \frac{dR}{ds} = -\cos\delta$

Having defined the relationship between s and R, we now can define the state equations

Heading error, δ

Eq. 2 $\quad \frac{d\delta}{ds} = \frac{\sin\delta}{R} + \kappa$

Eq. 3 $\quad \frac{d\delta}{dR} = -\frac{\tan\delta}{R} - \kappa\sec\delta$

Flight path angle, γ

Eq. 4 $\quad \frac{d\gamma}{dR} = -\kappa\sec\delta$

Defining the Change in Velocity, $\frac{dV}{dt}$

- The acceleration affecting missile velocity are summed to described total change in missile velocity as function of time
- One wishes to minimize the acceleration which is along the velocity vector, so Eq. 5 is rewritten do describe acceleration parallel and perpendicular to the velocity vector
- Eq. 6 describes the acceleration along the velocity vector. This is the acceleration to be minimized
- Eq. 7 is the acceleration perpendicular to the velocity vector. This, in essence, is the control
- Note that the N and A in Eq. 6 and Eq. 7 are accelerations, thus they are multiplied by W/G to convert the force equations to accelerations

- Missile accelerations are defined as

Eq. 5 $\quad \frac{dV}{dt} = \vec{F} + \vec{A} + \vec{N} + \vec{G}$

Rewrite the equation in terms of acceleration along and perpendicular to the velocity vector

Eq. 6 $\quad \frac{dV}{dt} = (F - A)\cos\alpha \; - N\sin\alpha \; - G\sin\gamma$

Eq. 7 $\quad V\frac{d\gamma}{dt} = (F - A)\sin\alpha + N\cos\alpha \; - G\cos\gamma$

One now describes the normal acceleration and the axial (zero-lift) acceleration can be written in term of aerodynamic coefficients and dynamic pressure.

Eq. 8 $\quad C_A = \left(\frac{W}{G}\right)\frac{A}{Q\,S_{Ref}} = \left(\frac{W}{G}\right)\frac{A}{\frac{1}{2}\rho V^2 S_{Ref}}$

Eq. 9 $\quad C_N \cong C_{N_\alpha}\,\alpha \; = \left(\frac{W}{G}\right)\frac{N\,\alpha}{Q\,S_{Ref}} = \left(\frac{W}{G}\right)\frac{N\,\alpha}{\frac{1}{2}\rho V^2 S_{Ref}}$

Describing the Control, κ

- Rearranging Eq. 8 and Eq 9 to solve for N and A brings to light the fact that there are a lot of constants in each equation. For simplicity moving forward, we gather all the constants into the terms k_A and k_N. We also define, b, a near constant parameter which gathers some of the second order terms into a single value
- Define:

$k_A = \frac{1}{2}\left(\frac{G}{W}\right)C_A\,S_{Ref}$

$k_N = \frac{1}{2}\left(\frac{G}{W}\right)C_{N_\alpha}\,S_{Ref}$

$b = \frac{F}{k_N\,\rho\,V^2} + 1 \; - \frac{k_A}{k_N}$

- Eq. 8 and Eq. 9 are rearranged to describe N and A in terms of aerodynamic properties

Eq. 10 $\quad A = k_A\,\rho\,V^2$

Eq. 11 $\quad N = k_N\,\rho\,V^2\,\alpha$

Substituting Eq. 10 and Eq. 11 into Eq. 7, and using small angle approximations for α yields

Eq. 12 $\quad V\frac{d\gamma}{dt} = (F - k_A\rho\,V^2)\frac{N}{k_N\,\rho V^2} + N \; - G\cos\gamma$

Eq. 13 $\quad V\frac{d\gamma}{dt} = N\left(\frac{F}{k_N\,\rho V^2} + 1 - k_A/k_N\right) - G\cos\gamma$

One divides Eq. 12 by V^2

Eq. 14 $\quad \frac{1}{V}\frac{d\gamma}{dt} = \frac{N}{V^2}\left(\frac{F}{k_N\,\rho V^2} + 1 - k_A/k_N\right) - \frac{G}{V^2}\cos\gamma$

Eq. 15 $\quad \kappa = \frac{N}{V^2}b \; - \frac{G}{V^2}\cos\gamma$

Eq. 16 $\quad N = \frac{1}{b}(V^2\,\kappa + G\cos\gamma)$

A Brief Recap

- Eq. 15 provides the definition of the control – the means in which the instantaneous curvature of the missile is controlled,

$$\kappa = \frac{N}{V^2}b \; - \frac{G}{V^2}\cos\gamma$$

- This equation can be rearranged to describe the normal acceleration

$$N = \frac{1}{b}(V^2\,\kappa + G\cos\gamma)$$

- Where

$$b = \frac{F}{k_N\,\rho\,V^2} + \left(1 \; - \frac{k_A}{k_N}\right)$$

What is the Magnitude of b?

- We have defined $b = \frac{F}{k_N\,\rho\,V^2} + \left(1 - \frac{k_A}{k_N}\right)$
- It can be assumed that $b \approx 1$
  - The table gives the rationale for this approximation

Term		Rationale	
Term	Rationale	Term	Rationale
$\frac{F}{k_N\,\rho\,V^2} \approx 0$	$F \ll \rho V^2$ unless the missile has significant rocket thrust	$\frac{F}{k_N\,\rho\,V^2} \approx 0$	$F \ll \rho V^2$ unless the missile has significant rocket thrust
$1 - \frac{k_A}{k_N} \approx 1$	$\frac{k_A}{k_N} \ll 0.05$ for missiles with good reaction capability	$1 - \frac{k_A}{k_N} \approx 1$	$\frac{k_A}{k_N} \ll 0.05$ for missiles with good reaction capability
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$1 - \frac{k_A}{k_N} \approx 1$	$\frac{k_A}{k_N} \ll 0.05$ for missiles with good reaction capability	$1 - \frac{k_A}{k_N} \approx 1$	$\frac{k_A}{k_N} \ll 0.05$ for missiles with good reaction capability

- One can also consider using this term as the value of $\frac{k_A}{k_N}$ is known by the guidance system designer

Describing the Change in Speed

- Dividing Eq. 12 and Eq. 13 by V^2 means both the acceleration along the velocity vector and perpendicular to the velocity vector are divided by V^2
- Making note of the relationship below, we can define rate of change of the speed of the missile with respect to the path length, s

$$\frac{1}{V^2}\frac{dV}{dt} = \frac{1}{V}\frac{dV}{ds} = \frac{d}{ds}[Log\ V]$$

- Eq. 19 accurately represents the rate of change of speed of the missile, but it can be rewritten in a manner which groups similar (constant or near constant) terms together, making this problem easier to understand.

Remember

Eq. 6 $\quad \frac{dV}{dt} = (F - A)\cos\alpha - N\sin\alpha - G\sin\gamma$

One divides Eq. 6 by V^2, just as was done for Eq. 7, and uses small angle approximations for α

Eq. 17 $\quad \frac{1}{V^2}\frac{dV}{dt} = \frac{1}{V^2}(F - A) - N\,\alpha - G\sin\gamma$

Making use of the definitions k_A and k_N

Eq. 18 $\quad \frac{d}{ds}[Log\ V] = \frac{F - G\sin\gamma}{V^2} - k_A\,\rho - \frac{N^2}{k_N\,\rho\,V^4}$

Using the definition of the normal acceleration from Eq. 16, one has

Eq. 19 $\quad \frac{d}{ds}[Log\ V] = \frac{F - G\sin\gamma}{V^2} - k_A\,\rho - \frac{(\kappa + G/V^2\cos\gamma)^2}{k_N\,\rho\,b^2}$

The Unabridged Cost Function, J_1

- Let's define some that allow us to group the contributing sources of missile acceleration more appropriately:

$$\chi = \frac{G}{V^2}\cos\gamma$$

$$\omega_0^2 = \frac{1}{2}k_A\,k_N\,b^2\,\rho^2$$

$$\omega^2 = \omega_0^2 - \frac{1}{2}\chi^2 - \frac{k_A\,b^2\,\rho\,[F - G\sin\gamma]}{2V^2}$$

- All of the above equations are a function of either V, ρ, or both V and ρ.
- There are multiple ways by which this hurdle can be overcome for the cost function J_1 to be optimized via closed form solution. The various methods are discussed next
- Note all methods arrive at the same result

Using the terms that are defined to the right, one rewrites Eq. 19

Eq. 20 $\quad \frac{d}{ds}[Log\ V] = -\frac{1}{k_N\,b^2\,\rho}(2\,\omega^2 + \kappa^2 + 2\,\chi\,\kappa)$

Using the relationship, $\kappa = \frac{d\gamma}{ds}$, Eq. 20 becomes

Eq. 21 $\quad \frac{d}{d\gamma}[Log\ V] = -\frac{1}{k_N\,b^2\,\rho}\left(2\frac{\omega^2}{\kappa} + \kappa + 2\,\chi\right)$

One can solve for the speed as a function of path length by noting

Eq. 21 $\quad V_f = V_0\,e^{-J_1}$

Where

Eq. 22 $\quad J_1 = \int_{\gamma_0}^{\gamma_f} \frac{2}{k_N\,b^2\rho}\left(\frac{\omega^2}{\kappa} + \frac{1}{2}\kappa + \chi\right)\,d\gamma$

Simplifying J_1, Method 1

- To optimize our trajectory, we must optimize the following cost function

Eq. 22 $\quad J_1 = \int_{\gamma_0}^{\gamma_f} \frac{2}{k_N\,b^2\rho}\left(\frac{\omega^2}{\kappa} + \frac{1}{2}\kappa + \chi\right)\,d\gamma$

- Note that b and k_N can be treated as constants
- Non-constants ρ and χ are replaced by their average value, $\bar{\rho}$ and $\bar{\chi}$ yielding

Eq. 23a $\quad J_1 = \frac{2}{k_N\,b^2\bar{\rho}}\left[\int_{\gamma_0}^{\gamma_f}\left(\frac{\omega^2}{\kappa} + \frac{1}{2}\kappa\right)\,d\gamma + \bar{\chi}\,(\gamma_f - \gamma_0)\right]$

- Thus, our cost function is now

Eq. 23b $\quad J = \int_{\gamma_0}^{\gamma_f}\left(\frac{\omega^2}{\kappa} + \frac{1}{2}\kappa\right)\,d\gamma$

- Finally, we replace ω with its average value, $\bar{\omega}$, which is only an "average" rather than an absolute due to $\bar{\rho}$) resulting in the our approximate cost function

Eq. 23c $\quad J = \int_{\gamma_0}^{\gamma_f}\left(\frac{\bar{\omega}^2}{\kappa} + \frac{1}{2}\kappa\right)\,d\gamma \quad = \quad \int_0^{R_0}\left(\bar{\omega}^2 + \frac{1}{2}\kappa^2\right)\sec\delta\ dR$

Simplifying J_1, Method 2

- To optimize our trajectory, we must optimize the following cost function

Eq. 22 $\quad J_1 = \int_{\gamma_0}^{\gamma_f} \frac{2}{k_N\,b^2\rho}\left(\frac{\omega^2}{\kappa} + \frac{1}{2}\kappa + \chi\right)\,d\gamma$

- Note that b and k_N can be treated as constants
- Non-constants ρ is replaced by its average value, $\bar{\rho}$
- Disregard the effect of gravity, and assume rocket thrust is small

Eq. 23a $\quad \bar{\omega}^2 = \bar{\omega}_0^2 - \frac{1}{2}\chi^2 - \frac{k_A\,b^2\,\bar{\rho}\,[F - G\sin\gamma]}{2V^2} \quad = \quad \omega_0^2$

Eq. 23b $\quad \bar{\omega}^2 = \bar{\omega}_0^2 = \frac{1}{2}k_A\,k_N\,b^2\,\rho^2$

- Thus, the cost function, J, becomes

Eq. 23c $\quad J = \int_0^{R_0}\left(k_A\rho + \frac{\kappa^2}{k_N\,b^2\bar{\rho}}\right)\sec\delta\,dR \quad = \quad \int_0^{R_0}\left(\bar{\omega}^2 + \frac{1}{2}\kappa^2\right)\sec\delta\,dR$

- But is it unrealistic to disregard gravity?
  - Not if there is no late maneuver penalty and there is a final flight path angle constraint

Setting Up the Pontryagin Equations

- The cost function is also the Lagrange cost of the system, thus Eq. 24 is often written as such

$$J = \int_0^{R_0} L\,dR$$

Where

$$L = \left(\bar{\omega}^2 + \frac{1}{2}\kappa^2\right)\sec\delta$$

Each of the two terms on the cost function represent a unique penalty:

$\bar{\omega}$ penalizes long trajectories (zero-lift drag)

$\frac{1}{2}\kappa^2$ penalizes excessive curvature (induced drag)

The letter "L" is used as the integrand represents the "Lagrange cost" of the system

- The cost function which is an approximation of the true cost function, but will be used in developing the optimal control is

Eq. 24 $\quad J = \int_0^{R_0}\left(\bar{\omega}^2 + \frac{1}{2}\kappa^2\right)\sec\delta\,dR$

Having defined the cost (above), one now has everything required to use of the Pontryagin Maximum Principle

State equations (Eq. 3 and 4) which were defined earlier

Eq. 3 $\quad \frac{d\delta}{dR} = -\frac{\tan\delta}{R} - \kappa\sec\delta$

Eq. 4 $\quad \frac{d\gamma}{dR} = -\kappa\sec\delta$

And the Hamiltonian of the system:

Eq. 25 $\quad H = L + \lambda_1\frac{d\delta}{dR} + \lambda_2\frac{d\gamma}{dR}$

The Optimum Value of κ

- The Pontryagin Maximum Principle requires 1 costate equation for each state variable in the system
- If there is no requirement for a prescribed final flight path angle, there is no need to include γ as a state variable, or the $\frac{d\gamma}{dR}$ state equation.
- The adjoint equations are determined via the relationships

$$\frac{d\lambda_1}{dR} = -\frac{\partial H}{\partial \delta}$$
$$\frac{d\lambda_2}{dR} = -\frac{\partial H}{\partial \gamma}$$

The Hamiltonian's full form is

Eq. 25 $$H = \left(\bar{\omega}^2 + \frac{1}{2}\kappa^2\right)\sec\delta - \lambda_1 \frac{\tan\delta}{R} - \kappa\sec\delta\,(\lambda_1 + \lambda_2)$$

The costate equations, as defined by their relationship to the Hamiltonian, are

Eq. 26 $$\frac{d\lambda_1}{dR} = -\left(\bar{\omega}^2 + \frac{1}{2}\kappa^2\right)\tan\delta\sec\delta + \lambda_1\frac{\sec^2\delta}{R} + \kappa\tan\delta\sec\delta\,(\lambda_1 + \lambda_2)$$

Eq. 27 $$\frac{d\lambda_2}{dR} = 0$$

Finally, the last condition of Pontryagin's Principle can be introduced – the equation for optimal control

Eq. 28 $$\frac{dH}{d\kappa} = 0 = \kappa\sec\delta - (\lambda_1 + \lambda_2)\sec\delta$$

Eq. 29 $$\kappa = \lambda_1 + \lambda_2$$

Solving for Costates λ_1 and λ_2

- The condition for optimal control has been defined

$$\kappa = \lambda_1 + \lambda_2$$

- However, the two costates are still undefined – all that is known is the costate equations of motion (Eq. 26 and Eq. 27)
- The solution for the second costate is readily accessible from Eq. 27

$$\lambda_2 = c$$

- The solution to the first costate is far more complicated, and is the heart of this optimization problem

Substituting the known value of λ_2 into the previous equation, only one unknown value remains - λ_1

Eq. 30 $$\kappa = \lambda_1 + c$$

Now, Eq. 30 is substituted into Eq. 26 in an attempt to solve for the remaining unknown

Eq. 31 $$\frac{d\kappa}{dR} = \left[\left(\frac{1}{2}\kappa^2 - \bar{\omega}^2\right)\sin\delta + \frac{\kappa - c}{R}\right]\frac{1}{1-\sin^2\delta}$$

Disregarding the third order terms ($\kappa^2\sin\delta$), and assuming $\frac{1}{1-\sin^2\delta} \approx 1$, the equation is simplified without a tremendous depreciation in accuracy

Eq. 32 $$\frac{d\kappa}{dR} = -\bar{\omega}^2\sin\delta + \frac{\kappa - c}{R}$$

The Differential Equation for the Optimal Trajectory

- The second order differential equation in Eq. 37 is the equation that will provide the optimal control and corresponding trajectory

$$\frac{d^2\xi}{dR^2} - \frac{2}{R}\frac{d\xi}{dR} - \bar{\omega}^2\xi = -c$$

- This type of equation (degenerate hypergeometric differential equation) is solved using modified Bessel functions of the first kind

$$\xi = \frac{c}{\bar{\omega}^2} + C_1\,(\bar{\omega}^2 R)^a\, I_a(z) + C_1\,(\bar{\omega}^2 R)^a\, I_{-a}(z)$$

- Where $I_a(z)$ is the modified Bessel function of order "a" and argument "z" with

$$a = 3/2$$
$$z = \bar{\omega}R$$

- While Bessel functions are fairly "unfriendly" with which to work, the ± 3/2 order modified Bessel functions can be represented with sinh and cosh functions

A couple of slick transformations will make Eq. 32 even more tractable

Define the instantaneous miss, ξ, and its derivative (with help from Eq. 3)

Eq. 33 $$\xi = R\sin\delta$$

Eq. 34 $$\frac{d\xi}{dR} = -\kappa R$$

Thus, Eq. 32 becomes

Eq. 35 $$\frac{d}{dR}\left[-\frac{1}{R}\frac{d\xi}{dR}\right] + \frac{1}{R^2}\frac{d\xi}{dR} = -\bar{\omega}^2\sin\delta - \frac{c}{R}$$

Eq. 36 $$\frac{1}{R^2}\frac{d\xi}{dR} - \frac{1}{R}\frac{d^2\xi}{dR^2} + \frac{1}{R^2}\frac{d\xi}{dR} = -\bar{\omega}^2\sin\delta - \frac{c}{R}$$

Eq. 37 $$\frac{d^2\xi}{dR^2} - \frac{2}{R}\frac{d\xi}{dR} - \bar{\omega}^2\xi = c$$

Solution to the Differential Equation

- Eq. 38 is the solution to the differential equation, but it has not been evaluated at the boundary points as of yet
- Noting at $R = 0$

$$\xi(R=0) = \xi_f = 0$$
$$\gamma(R=0) = \gamma_f$$

- It is a side note that the terminal transversality condition for this problem is not needed as it states that the final unknown values of $\boldsymbol{\lambda}(R=0)$ is simply equal to another unknown value, $\boldsymbol{\nu}$:

$$\lambda_1 = \frac{\partial}{\partial\delta}\left[\nu_2(\delta - 0)\right] = \nu_1$$
$$\lambda_2 = \frac{\partial}{\partial\delta}\left[\nu_2(\gamma - \gamma_f)\right] = \nu_2$$

The solution to the differential equation in Eq. 37 is

Eq. 38 $$\xi = \frac{c}{\bar{\omega}} + C_1(z\cosh z - \sinh z) + C_2(z\sinh(z) - \cosh z)$$

Since the miss distance ($\xi = R\sin\delta$) must be zero at intercept, one can evaluate Eq. 38 at $R = 0$ and substitute the expression for c back into Eq. 38

Eq. 39 $$c = \bar{\omega}^2 C_2$$

Eq. 40 $$\xi = C_1(z\cosh z - \sinh z) + C_2(z\sinh(z) - \cosh z + 1)$$

One makes use of the relationship between ξ and κ as defined in Eq. 34 – differentiating Eq. 40 to arrive at an expression of the control, κ

Eq. 41 $$\kappa = -C_1\bar{\omega}^2\sinh z - C_2\bar{\omega}^2\cosh z$$

Now the other boundary condition is used to provide the second equation to solve for the two unknowns

Solving for C_1 and C_2

- Eq. 40 and Eq. 43 can be used to solve the unknowns C_1 and C_2 by evaluating the two equations at $R = R_0$ and $R = 0$
- Using the notation

$$\begin{bmatrix}\xi \\ (\gamma - \gamma_f)/\bar{\omega}\end{bmatrix} = M\begin{bmatrix}C_1 \\ C_2\end{bmatrix}$$

- Where

$$M = \begin{bmatrix}(z\cosh z - \sinh z) & z\sinh(z) - \cosh z + 1 \\ \cosh z - 1 & \sinh z\end{bmatrix}$$

- Constants C_1 and C_2 are found to be

$$C_1 = \frac{\sinh z\,\xi + (\cosh z - 1 - z\sinh z)(\gamma - \gamma_f)/\bar{\omega}}{z\sinh z - 2(\cosh z - 1)}$$

$$C_1 = \frac{\bar{\omega}^2(\cosh z - 1)\xi + \bar{\omega}(z - \sinh z)(\gamma - \gamma_f)/\bar{\omega}}{z\sinh z - 2(\cosh z - 1)}$$

It is known from the second state equation that

Eq. 4 $$\frac{d\gamma}{dR} = -\kappa\sec\delta$$

One approximates this to be

Eq. 42 $$\frac{d\gamma}{dR} = -\kappa$$

Thus,

Eq. 43 $$\gamma - \gamma_f = -\int_0^R \kappa\; dR$$

Eq. 44 $$\frac{\gamma - \gamma_f}{\omega} = C_1(\cosh z - 1) + C_2\sinh z$$

Substituting the values of C_1 and C_2 (see the notes to left) into Eq. 41 gives the closed form solution to the control, κ

Eq. 45 $$\kappa = \frac{\bar{\omega}(z - \sinh z)(\gamma - \gamma_f) + \bar{\omega}^2(\cosh z - 1)\xi}{z\sinh z - 2(\cosh z - 1)}$$

The Conventional Form of Kappa Guidance

- Remember the shorthand notation that was introduced earlier:

$$z = \bar{\omega}R$$

- It is advantageous to make κ a function of independent variable, z, rather than constant $\bar{\omega}$ and independent variable, R. This is accomplished by multiplying κ by R/R and converting ξ back into $R\sin\delta$
- Once the guidance gains have been transformed such that $\kappa = \kappa(z)$, the guidance equation can be broken into a heading error term, a trajectory shaping term, and the someone less interesting gravity compensation terms

The acceleration command is defined by Eq. 16, where we assume $b \approx 1$,

Eq. 46 $$N = \frac{V^2}{R}\,\kappa + G\cos\gamma$$

In order to reduce the number of variables by one (converting $\bar{\omega}$ terms into z terms), κ is multiplied by R/R

Eq. 47 $$\mathrm{N} = \frac{\mathrm{V}^2}{\mathrm{R}}\,(\kappa R) + G\cos\gamma$$

And κR becomes

Eq. 48 $$\kappa R = \frac{(z\sinh z - z^2)(\gamma - \gamma_f) - z^2(\cosh z - 1)\sin\delta}{z\sinh z - 2(\cosh z - 1)}$$

Kappa Guidance Gains

- Eq. 46 and Eq. 47 are the kappa guidance gains which are used to provide the optimal trajectory to a given intercept point, considering both zero lift drag and induced drag
- As you'll see on the next slide, the guidance gains are function of z - such that even for a constant $\bar{\omega}$, the guidance gains change as the missile approaches the intercept point
- BUT, for the special case where $\bar{\omega} = 0$, or $R = 0$, the kappa guidance gains are equal to the guidance gains developed when zero lift drag is not considered
  - Thus, OG to a PIP with Trajectory Shaping can be considered a special case of Kappa Guidance

- Separating the guidance command into the three components (heading error, trajectory shaping, and gravity compensation) yields

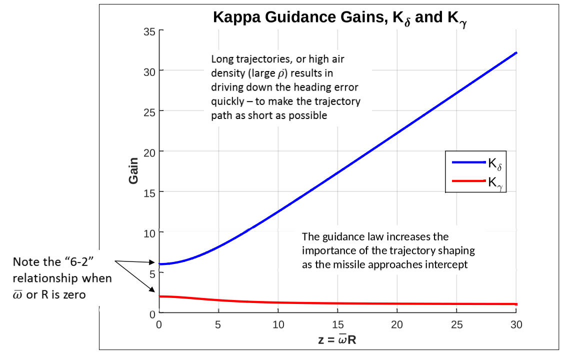
Eq. 45 $$\mathrm{N} = = -\frac{V^2}{R}\left(K_\delta \sin\delta - K_\gamma(\gamma - \gamma_f)\right) + G\cos\gamma$$

Where:

Eq. 46 $$K_\delta = \frac{z^2(\cosh z - 1)}{2(\cosh z - 1) - z\sinh z}$$

Eq. 47 $$K_\gamma = \frac{(z\sinh z - z^2)}{2(\cosh z - 1) - z\sinh z}$$

Kappa Guidance Gains, Illustrated



The Kappa Guidance Trajectory

- Kappa guidance uses $\bar{\omega}$ to control path length, often times limiting apogee