Homework 6 - Aidan Sharpe

Exercise 1

Consider a solenoid containing two coaxial magnetic rods of radii a and b and permeabilities $\mu_1 = 2\mu_0$ and $\mu_2 = 3\mu_0$. If the solenoid has n turns every d meters along the axis and carries a steady current, I.

a)

Find $\vec{H}$ inside the first rod, between the first and second rod, and outside the second rod.

For a solenoid regardless of magnetic materials:

$$\vec{H} = \frac{In}{d}\hat{z}$$

So for all regions:

$$\vec{H} = \frac{In}{d}\hat{z}$$

b)

Find $\vec{B}$ inside the three regions.

$\vec{B}$ relies on magnetic material properties:

$$\vec{B} = \mu\vec{H}$$

Therefore:

$$\vec{B} = \begin{cases} 2\mu_0\vec{H} & \text{In region 1} \\ 3\mu_0\vec{H} & \text{In region 2} \\ \mu_0\vec{H} & \text{In region 3} \end{cases}$$

c)

$\vec{M}$ in each region:

By definition:

$$\vec{M} = \chi_m\vec{H}$$

$$\chi_m = \mu_r - 1$$

Therefore:

$$\vec{M} = \begin{cases} \vec{H} & \text{In region 1} \\ 2\vec{H} & \text{In region 2} \\ 0 & \text{In region 3} \end{cases}$$

d)

$\vec{J}$ in each region:

By definition:

$$\vec{J}_m = \nabla \times \vec{M}$$

Since $\vec{M}$ is both conservative and solenoidal inside the solenoid, for all three regions:

$$\vec{J}_m = 0$$

Exercise 2

In a nonmagnetic medium:

$$E = 4\sin(2\pi \times 10^7 t - 0.8x)\hat{z}$$

a)

Find ε_r and η:

The general form:

$$\vec{E}(x,t) = E_0 \cos(\omega t - \beta x)\hat{z}$$

$$\beta = \omega\sqrt{\mu\varepsilon} = 0.8$$

$$0.8 = 2\pi \times 10^7 \sqrt{\mu_0 \varepsilon_0 \varepsilon_r}$$

$$\frac{\left(\frac{0.8}{2\pi \times 10^7}\right)^2}{\mu_0 \varepsilon_0} = \varepsilon_r$$

$$\boxed{\varepsilon_r = 14.566}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_r}}$$

$$\boxed{\eta = 98.275}$$

b)

Find the average poynting vector

$$\vec{P}_{\text{avg}} = \frac{E_0^2}{2\eta}\hat{x} = \frac{4^2}{2(98.275)}$$

$$\vec{P}_{\text{avg}} = 0.081$$

Exercise 3

$$E = 3\sin(2\pi \times 10^7 t - 0.4\pi x)\hat{y} + 4\sin(2\pi \times 10^7 t - 0.4\pi x)\hat{z}$$

b)

$$\varepsilon_r = \frac{\left(\frac{\beta}{\omega}\right)^2}{\mu_0\varepsilon_0} = \frac{\left(\frac{0.4\pi}{2\pi\times 10^7}\right)^2}{\mu_0\varepsilon_0}$$

$$\boxed{\varepsilon_r = 35.941}$$

a)

$$\lambda = \frac{v_p}{f}$$

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = 5\times 10^7$$

$$f = \frac{\omega}{2\pi} = 10^7$$

$$\boxed{\lambda = 5[\text{m}]}$$

c)

$$H = \frac{E_0}{\eta}\sin(\omega t - \beta x)\hat{z} + \frac{E_0'}{\eta}\sin(\omega t - \beta x)\hat{y}$$

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r}} = 62.85$$

$$\boxed{H = 0.0477\sin(2\pi\times 10^7 t - 0.4\pi x)\hat{z} + 0.0636\sin(2\pi\times 10^7 t - 0.4\pi x)\hat{y}}$$

Exercise 4

Prove that:

$$\hat{H}_y = -\frac{\hat{E}_x}{\eta_0}$$

Starting with a genering electric plane wave in the $-\hat{z}$ direction:

$$\vec{E}(z,t) = E_0\cos(\omega t + \beta_0 z)\hat{x}$$

We will use Faraday's Law of Induction to convert a measure in the $\vec{E}$-field to a value for the $\vec{B}$-field:

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

Converting from $\vec{B}$ to $\vec{H}$:

$$\nabla\times\vec{E} = -\mu\frac{\partial}{\partial t}\vec{H}$$

Assuming free space:

$$\mu = \mu_0$$

This gives the final form for Faraday's Law:

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial}{\partial t} \vec{H}$$

Evaluating the curl of $\vec{E}$:

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \|\vec{E}\| & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z} \|\vec{E}\| \hat{y} - \frac{\partial}{\partial y} \|\vec{E}\| \hat{z}$$

Since $\vec{E}$ only varies with respect to z and t this can be rewritten as:

$$\nabla \times \vec{E} = \frac{\partial}{\partial z} \|\vec{E}\| \hat{y}$$

Evaluate the partial derivative:

$$\nabla \times \vec{E} = -E_0 \beta_0 \sin(\omega t + \beta_0 z) \hat{y}$$

Back to Faraday's Law:

$$-E_0 \beta_0 \sin(\omega t + \beta_0 z) \hat{y} = -\mu_0 \frac{\partial}{\partial t} \vec{H}$$

Divide out by $-\mu_0$:

$$\frac{\partial}{\partial t} \vec{H} = \frac{E_0 \beta_0}{\mu_0} \sin(\omega t + \beta_0 z) \hat{y}$$

Expand out β_0:

$$\beta_0 = \omega \sqrt{\mu_0 \varepsilon_0}$$

$$\frac{\partial}{\partial t} \vec{H} = \frac{E_0 \omega \sqrt{\mu_0 \varepsilon_0}}{\mu_0} \sin(\omega t + \beta_0 z) dt \hat{y}$$

Integrate both sides with respect to t:

$$\vec{H}(z,t) = \frac{E_0 \omega \sqrt{\mu_0 \varepsilon_0}}{\mu_0} \int \sin(\omega t + \beta_0 z) dt \hat{y}$$

Evaluate the integral:

$$\vec{H}(z,t) = -\frac{E_0 \omega \sqrt{\mu_0 \varepsilon_0}}{\mu_0 \omega} \cos(\omega t + \beta_0 z) \hat{y}$$

Simplifying the fraction:

$$\vec{H}(z,t) = -E_0\sqrt{\frac{\varepsilon_0}{\mu_0}}\cos(\omega t + \beta_0 z)\hat{y}$$

Using the definition of η_0:

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$\vec{H}(z,t) = -\frac{E_0}{\eta_0}\cos(\omega t + \beta z)\hat{y}$$

$$\therefore \vec{H}(z,t) = -\frac{\|\vec{E}(z,t)\|}{\eta_0}\hat{y}$$