Introduction to Track Filtering (State Estimation)

Gregg Bock

What is Track Filtering?

- The process in which measurements from a single object are used to estimate the kinematic states of that object has many names. Two of the more common names for this process are
  - **State estimation**
    - This term is the most accurate description of the process
    - It can apply to concepts in robotics, servos, radar, etc..
  - **Track filtering (or simply "filtering")**
    - This term is often used for interpreting observations collected from radar/sensor systems
- For our purposes, the names can be used interchangeably
- Many good books can be found on the subject of track filtering. Some suggested texts for an introduction to filtering include:
  - Bar-Shalom, *Estimation with Applications to Tracking and Navigation: Theory Algorithms and Software*. 2001
  - Gelb, *Applied Optimal Estimation*. MIT Press, 1974 (out of print, but easy to find in the library)

Focus of Track Filtering in This Course

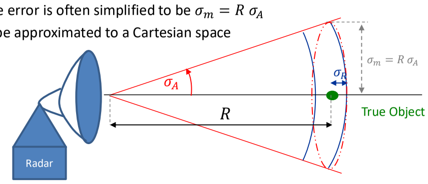
- Discussion of track filtering in this course is meant to provide a basic, yet broad, introduction to track filtering and its uses to a weapon system designer
  - Filter designs to meet specific weapon system requirements
  - The importance of the filter design in error budget allocation and analysis
- State estimation (of which track filtering is a subset) is a very broad subject which consistently garners tremendous attention in mathematical circles as the concept of "the best" state estimator is always hotly debated

Track Filtering

- All combat systems use track filtering as a means to process sensor measurements of an object and form an estimate of the object's kinematic states
- Track filtering in the combat system became been necessary with the introduction of sensor measurement systems (e.g. radar, sonar) in the early 20th century
  - Sensors provide discrete estimates of object data (measurements) over time
    - Position (range, bearing, elevation)
    - Range rate with respect to the radar (Doppler)
  - Successive position measurements over time are "strung together" to estimate kinematic states
    - Position
    - Velocity
    - Acceleration
    - Jerk
    - …

Why is a Track Filter Necessary?

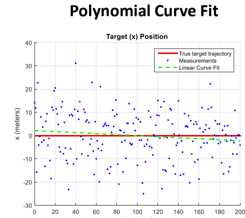
- Measuring an object using an electromagnetic, electo-optical, or audio sensor results in an observation of that object with error
- Most sensors have the following error characteristics
  - Errors in along the line-of-sight from sensor to object are independent of range
  - Errors perpendicular to the line-of-sight from sensor to object are range dependent
    - Typically can be described as linearly increasing with respect to range
    - i.e. angle error is constant
- In fire control systems, it is typical for the angle error to be on the order a few milliradians
  - Thus, the crossrange error is often simplified to be $\sigma_m = R\,\sigma_A$
  - And the errors can be approximated to a Cartesian space



A Filter's Relationship to Data Analysis Techniques

- In data analysis, one uses a polynomial to find a curve that best fits a set of data (linear, quadratic, etc.)
- In track filtering, the data is provided one observation (measurement) at a time, and a recursive algorithm is used to correct the filter state estimate after each new observation is considered
- Unlike the polynomial curve fit, the filter can consider an error to the model assumption in addition to the error in the measurement

Polynomial Curve Fit



Target Filter

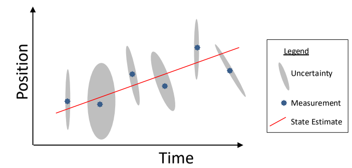
Observation

COMPONENTS OF A STATE ESTIMATOR

Observation

Components of State Estimation

- State estimators require multiple observations of the object whose states are to be estimated
  - An observation is called a measurement, z
  - Each measurement is collected at a discrete time, t
  - A description of the accuracy of the measurement at time, t, is either provided or assumed. For example, measurement z has the following characteristics:
    - $R = \mathcal{N}(\mu, \sigma_m^2)$
    - R is a normally distributed error with a variance of σ_m^2 and a mean of μ
- Therefore, the k^{th} observation of an object is described by the following:
  - Time of the measurement, t_k
  - Measurement, z_k
  - Measurement uncertainty, R_k



Initialization

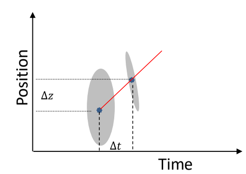
COMPONENTS OF A STATE ESTIMATOR

Initialization

Components of State Estimation

- A radar measures the position of a given object assumed to be moving in a straight line
- A single observation is made at $time = 0$ (observation number $k = 0$)
  - For the $k = 0$ observation,
    - Position $\hat{x}_{0|0} = z_0$
    - Velocity $\dot{\hat{x}}_{0|0} = 0$

    The $x_{k|k}$ notation is used to indicate the state estimate at k, was made based upon data from observation k
  - Since there is only one observation, the observation is the best state estimate
- When a second observation occurs, an estimate of other kinematic states is made
  - From the $k = 1$ observation, it is concluded that
    - $\dot{\hat{x}}_{1|1} = \frac{\Delta z}{\Delta t} = \frac{(z_1 - z_0)}{t_1 - t_0}$



Prediction

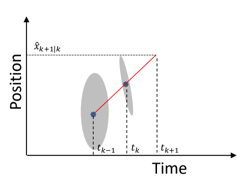
Components of State Estimation

- Given an estimate of position ($\hat{x}_k$) and velocity ($\dot{\hat{x}}_k$), an estimate of the states at any time in the future is possible

$$\hat{x}_{k+1|k} = \hat{x}_{k|k} + \dot{\hat{x}}_{k|k}\,(t_{k+1} - t_k)$$

$$\dot{\hat{x}}_{k+1|k} = \dot{\hat{x}}_{k|k}$$

The $x_{k+1|k}$ notation is used to indicate the state estimate at $k+1$, was made based upon data from observation k

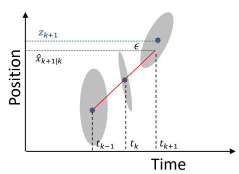


Correction / Update

COMPONENTS OF A STATE ESTIMATOR

Correction / Update, Residual Error
Components of State Estimation

- Predicting the state estimates to the time of the next observation leads to two estimates of position at time, t_{k+1}
  - Prediction: $\hat{x}_{k+1|k}$
  - Measurement: z_{k+1}
- There difference between the two estimates is called the residual error
  - $\epsilon = z_{k+1} - \hat{x}_{k+1|k}$

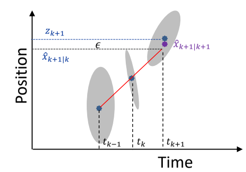


Correction / Update
Components of State Estimation

- The filter corrects (updates) the state estimates given the new information (z_{k+1}) using a weighted average of the residual based upon confidence in each estimate
  - The weighted average is called the gain
- Once a gain has been selected for each state, the estimate of the states can be updated
  - $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_1\left(z_{k+1} - \hat{x}_{k+1|k}\right)$
  - $\dot{\hat{x}}_{k+1|k+1} = \dot{\hat{x}}_{k+1|k} + K_2\left(z_{k+1} - \hat{x}_{k+1|k}\right)$

Updated state — Predicted state — Gain — Residual error

The position gain, K_1, has a value which is typically between 0 and 1



Equations of the Recursive Filter
Summary

Prediction:

$$\hat{x}_{k+1|k} = \hat{x}_{k|k} + \dot{\hat{x}}_{k|k}\left(t_{k+1} - t_k\right)$$

$$\dot{\hat{x}}_{k+1|k} = \dot{\hat{x}}_{k|k}$$

Determine Gains:

$$K_1 = f(\cdot),\quad K_2 = f(\cdot)$$

Update:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_1\left(z_{k+1} - \hat{x}_{k+1|k}\right)$$
$$\dot{\hat{x}}_{k+1|k+1} = \dot{\hat{x}}_{k+1|k} + K_2\left(z_{k+1} - \hat{x}_{k+1|k}\right)$$

Described in matrix form, $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$, which may be used as shorthand to represent gains

The Key to Any Filter's Performance is the Gain Selection Methodology

THE KALMAN FILTER

A Little Filter History

- Prior to the 1960s, the selection of filter gains (K) for target tracking applications were not always derived using mathematical rigor
  - Some might even say the gains were chosen with artistic flair
  - Mathematicians knew how to determine stable filter gains (to avoid instability in recursive algorithms), but optimality was not considered in gain selection
- The 2 state filter was often referenced as either the $f - g$ filter, the $g - h$ filter or (the now most common name) the $\alpha - \beta$ filter
- The three state filter is commonly referred to as an $\alpha - \beta - \gamma$ filter

Gain	Reference to K Matrix	Stability Criteria
Position State	$\alpha = \mathrm{K}(1)$	$0 < \alpha < 2$
Velocity State	$\beta = \mathrm{K}(2)\,T$	$0 < \beta,\ \ \beta < 4 - 2\alpha$
Acceleration State	$\gamma = \frac{1}{2}\mathrm{K}(3)\,T^2$	$0 < \gamma,\ \ \gamma < \frac{4\alpha\beta}{2-\alpha}$

The Modern Age of Track Filtering
1960s

- Remarkable growth in developing mathematical approaches for solving track filtering problems occurred in the late 1950s and early 1960s
- Benedict & Bordner derived a relationship between the gains of a 2 state filter ($\alpha - \beta$ filter) which is known today as the Benedict / Bordner (B&B) relationship
  - It satisfied the requirement for track filter stability
  - It quickly was adopted as the "go to" track filter gain relationship for system design
- The introduction of the Kalman filter provides a means to select filter gains based upon mathematical principles of optimization
  - The filter provided a mathematical model for a recursive filter which minimized the total error of the estimator
  - The error characteristics used in the filter assume both target dynamic model errors (often referred to as process noise) and measurement errors are Gaussian in nature

The Kalman Filter Allowed for the Selection of Filter Gains Using Optimization Theory in a Recursive Algorithm

Kalman Filter Fever

- The publication of the optimal filter approach suggested by Kalman and Bucy (often referred to simply as the Kalman filter) began the modern age of track filtering
- The invention of the Kalman filter, in conjunction with an increase in computer processing speed available to the user, meant the Kalman filter could be used in software programs for real time state estimate applications
  - NASA / Space program
  - DoD military applications
- The validity of the assumptions made in the derivation of the Kalman filter became (and still are) the source of many debates in the mathematical community

Kalman Filter Definitions

Parameter	Symbol	Definition	Dimensions
Measurement	z	An instance of observation by the radar	$(q) \times (1)$
Measurement variance	$R \equiv \sigma_m^2$	The variance of the measurement, $R = \mathcal{N}(0, \sigma_m^2)$	$(q) \times (q)$
Observation Matrix	H	Matrix which indicates which target state is measured by the radar	$(q) \times (n)$
Transition Matrix	φ	Matrix which defines how the target states (and the uncertainties) propagate between observations	$(q \times n) \times (q \times n)$
Target Model Error Intensity	W	Defines the error in the target dynamic model, $W = \mathcal{N}(0, W)$	$(1) \times (1)$
Process Noise Matrix	Q	Expresses how the error influences the estimates states between observations	$(q \times n) \times (q \times n)$
State estimate	$\hat{x}$	Estimate of the target kinematic states based upon radar measurements	$(q \times n) \times (1)$
State uncertainty	P	Quantitative description of the error in the state estimates	$(q \times n) \times (q \times n)$

n states per dimension (position, velocity, etc.)
q Number of dimensions measured

The Kalman Filter Equations
Matrix Form

Definitions

- 1-D, 2 state Kalman filter
  - Position, velocity in one dimension

Observation matrix, $H = [1\ 0]$

Transition Matrix, $\varphi = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$

State covariance, P, is a 2x2 matrix

Initialized to : $P_{0|0} = \begin{bmatrix} 1 & 1/\Delta T \\ 1/\Delta T & 2/\Delta T^2 \end{bmatrix} R$

Process noise, Q, is a 2x2 matrix

Kalman Equations

Prediction:

$\Delta t = t_k - t_{k-1}$

$\hat{x}_{k|k-1} = \varphi\, \hat{x}_{k-1|k-1}$

$P_{k|k-1} = \varphi\, P_{k-1|k-1}\, \varphi^T + Q_k$

Gain Computation:

$K = P_{k|k-1}\, H^T \left(H\, P_{k|k-1}\, H^T + R\right)^{-1}$

Update:

$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K\left(z_k - H\, \hat{x}_{k|k-1}\right)$

$P_{k|k} = (I - KH) P_{k|k-1} (I - KH)^T + KRK^T$

The Q Matrix Requires Further Discussion as Its Definition is Key to Gain Selection

Process Noise, Q
The Basics

- The process noise, Q, describes the uncertainty in estimated states due the error in the assumptions of the motion of the object being tracked
- Q consists of a dynamics error model (described in matrix form in this lecture) and an error intensity (W)
- Three of the canonical process noise models for the 2 state Kalman filter are
  - Discrete white noise acceleration (DWNA)
  - Continuous white noise acceleration (CWNA)
  - Velocity perturbation (B&B)

DWNA	CWNA	B&B
$Q = \begin{bmatrix} \frac{1}{4}\Delta T^4 & \frac{1}{2}\Delta T^3 \\ \frac{1}{2}\Delta T^3 & \Delta T^2 \end{bmatrix} W$	$Q = \begin{bmatrix} \frac{1}{3}\Delta T^3 & \frac{1}{2}\Delta T^2 \\ \frac{1}{2}\Delta T^2 & \Delta T \end{bmatrix} W$	$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} W$

Process Noise
Details of Error Intensity, W

- W always represents the intensity of the model uncertainty between update time t_k and t_{k+1},
  - W has a different interpretation (and magnitude) dependent upon the process noise model
  - We use different nomenclature to describe W based upon process noise model
- DWNA:
  - σ_w is meant to be a 1 σ value of the unmodeled acceleration, the filter designer often takes liberty with the magnitude of σ_w to get the desired performance
- CWNA:
  - $\tilde{q}$ is the integrated error over ΔT, computed by converting continuous process noise to discrete time process noise
  - The process noise intensity, $\tilde{q}$, is selected based upon the change in velocity over the time period ΔT, which is on the order of $\sqrt{\tilde{q}\Delta T}$
- B&B
  - σ_v is meant to be a 1 σ value of the perturbation of velocity

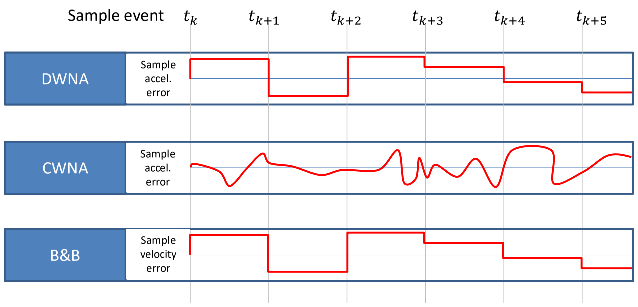
Process Noise
Target Dynamics Error Modeling

- The DWNA and B&B models follow a rather straightforward method of computing the expectation value (which is zero) and variance (which is not zero) of the process noise
- The CWNA model is a little more complex as one needs to determine the expectation of the continuous random error integrated over time period ΔT
  - See Bar-Shalom book for a complete derivation

Model Error	Error Definition	Q Matrix	W
DWNA	$Q = \begin{bmatrix} \frac{1}{2}\Delta T^2 \\ \Delta T \end{bmatrix} \sigma_w^2 \begin{bmatrix} \frac{1}{2}\Delta T^2 & \Delta T \end{bmatrix}$	$Q = \begin{bmatrix} \frac{1}{4}\Delta T^4 & \frac{1}{2}\Delta T^3 \\ \frac{1}{2}\Delta T^3 & \Delta T^2 \end{bmatrix} \sigma_w^2$	$W = \sigma_w^2$
CWNA	$Q = \int_0^{\zeta} \begin{bmatrix} \Delta T - \zeta \\ 1 \end{bmatrix} [\Delta T - \zeta \quad 1]\, \tilde{q}\, d\zeta$	$Q = \begin{bmatrix} \frac{1}{3}\Delta T^3 & \frac{1}{2}\Delta T^2 \\ \frac{1}{2}\Delta T^2 & \Delta T \end{bmatrix} \tilde{q}$	$W = \tilde{q}$
B&B	$Q = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma_v^2 [0 \quad 1]$	$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sigma_v^2$	$W = \sigma_v^2$

Illustration of Common Q Models

- DWNA and B&B errors are Gaussian , but errors in the target model only change at a sample event
- CWNA errors are also Gaussian, but the samples are continuous, regardless of sample event



Gain Relationships and Q Models

- Not only does each of the process noise models have a unique Q matrix, but each has a specific gain relationship between gains α and β

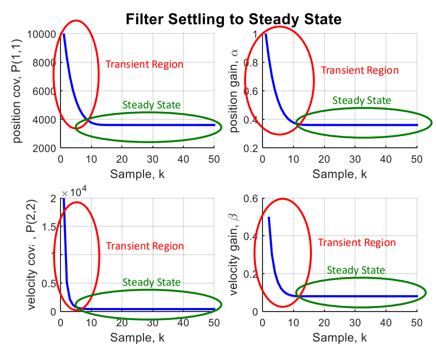
Model Error	Q Matrix	W Units	Gain Relationship
DWNA	$Q = \begin{bmatrix} \frac{1}{4}\Delta T^4 & \frac{1}{2}\Delta T^3 \\ \frac{1}{2}\Delta T^3 & \Delta T^2 \end{bmatrix} W$	$\frac{m^2}{s^4}$	$\beta = 2\left(1 - \sqrt{1-\alpha}\right)^2$
CWNA	$Q = \begin{bmatrix} \frac{1}{3}\Delta T^3 & \frac{1}{2}\Delta T^2 \\ \frac{1}{2}\Delta T^2 & \Delta T \end{bmatrix} W$	$\frac{m^2}{s^3}$	$\beta = 6 - 3\alpha - \sqrt{3\alpha^2 + 36(1-\alpha)}$
B&B	$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} W$	$\frac{m^2}{s^2}$	$\beta = \frac{\alpha^2}{2-\alpha}$

Gain Relationships are Dependent Upon the Type of Process Noise

Steady State…

- A filter will reach steady state if the following parameters are constant over time
  - Measurement variance, R
  - Update period, Δt
  - Target model error, Q
- When steady state is reached, the following filter traits are realized
  - Target covariance matrix, $P_{k|k} = P_{k+1|k+1}$
  - Gain matrix, $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta/\Delta T \end{bmatrix}$, is constant from update to update
- Note that:
  - $P_{k|k}$ can be separated into a steady state error due to measurements ($M_{k|k}$), and error due to target model error ($Y_{k|k}$)
  - $P_{k|k} = M_{k|k} + Y_{k|k}$
  - $Y_{k|k}$ is not discussed as frequently as $P_{k|k}$ and $M_{k|k}$

Illustration of Steady State



Steady State Errors for M and P

Model	Steady State $P_{k\|k}$	Steady State $M_{k\|k}$
DWNA	$P_{k\|k} = \begin{bmatrix} \alpha & \frac{\beta}{\Delta T} \\ \frac{\beta}{\Delta T} & \frac{\beta(2\alpha-\beta)}{2(1-\alpha)\Delta T^2} \end{bmatrix} \sigma_m^2$	$M_{k\|k} = \begin{bmatrix} 2\alpha^2 + \beta(2-3\alpha) & \beta(2\alpha-\beta)/\Delta T \\ \beta(2\alpha-\beta)/\Delta T & 2\beta^2/\Delta T^2 \end{bmatrix} \frac{\sigma_m^2}{u_0}$
CWNA		
B&B	$P_{k\|k} = \begin{bmatrix} \alpha & \frac{\beta}{\Delta T} \\ \frac{\beta}{\Delta T} & \frac{\alpha\beta}{(1-\alpha)\Delta T^2} \end{bmatrix} \sigma_m^2$	

Where:

$u_0 = \alpha(4 - 2\alpha - \beta)$

The P matrix contains all the error in the estimate of the states

The M matrix only contains the state estimate error resultant from the measurement error

Steady State Errors for Y

Model	Steady State $Y_{k\|k}$
DWNA	$Y_{k\|k} = \begin{bmatrix} 2(1-\alpha)^2\Delta T^4 & (2\alpha+\beta-2\alpha\beta)\Delta T^3 \\ (2\alpha+\beta-2\alpha\beta)\Delta T^3 & (2\alpha^2-3\alpha\beta+2\beta)\Delta T^2 \end{bmatrix} \frac{\sigma_w^2}{4\alpha\beta}$
CWNA	$Y_{k\|k} = \begin{bmatrix} 2(1-\alpha)^2(6-3\alpha-\beta)\Delta T^3 & (1-\alpha)(\beta(\beta-6)+6\alpha(2-\alpha))\Delta T^2 \\ (1-\alpha)(\beta(\beta-6)+6\alpha(2-\alpha))\Delta T^2 & u_3\Delta T \end{bmatrix} \frac{\tilde{q}}{6\beta u_0}$
B&B	$Y_{k\|k} = \begin{bmatrix} (2-\alpha)(1-\alpha)^2\Delta T^2 & (2\alpha-\beta-\alpha^2)(1-\alpha)\Delta T \\ (2\alpha-\beta-\alpha^2)(1-\alpha)\Delta T & \alpha^2(2-\alpha)+2\beta(1-\alpha) \end{bmatrix} \frac{\sigma_v^2}{\beta u_0}$

Where:

$u_0 = \alpha(4 - 2\alpha - \beta)$

$u_3 = 2(3\alpha^2(\beta - \alpha + 2) - \beta(\beta - 6)(1 - 2\alpha))$

The Y matrix only contains the state estimate error resultant from the process noise

Filter Optimization

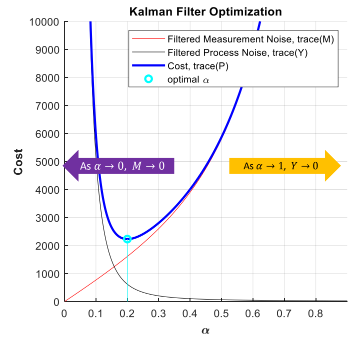
- The Gelb book (if you can find it) has a wonderful derivation of the Kalman filter gains
  - But please look elsewhere for a derivation if you prefer
  - There are many different derivations for the Kalman filter – all give the same answer
- The Kalman filter has a very specific optimization criterion (or cost function) providing a dynamically computed gain matrix, K
- The optimization criteria for the Kalman filter is to minimize the trace of the covariance matrix, P

  Minimizing $trace(P)$ is the same as minimizing $\sigma_p^2 + \sigma_v^2$ ← Note the difference in units

  Where $P = \begin{bmatrix} \sigma_p^2 & \sigma_{pv} \\ \sigma_{vp} & \sigma_v^2 \end{bmatrix}$,

  $\sigma_{pv} = \sigma_{vp}$ because the covariance matrix is symmetrical
- This optimization criteria yields the gain equation: $K = P_{k|k-1} H^T \left(H P_{k|k-1} H^T + R\right)^{-1}$

Illustration of Optimization
Kalman Filter Optimization - DWNA



DWNA Filter Model

Sensor:

σ_m = 100 m

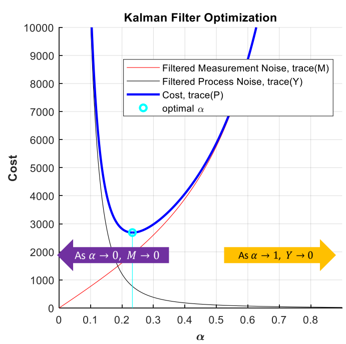
Δt = 0.5 sec

Filter:

σ_w = 10 m/s^2

Optimal α =0.2

Illustration of Optimization
Kalman Filter Optimization - CWNA



CWNA Filter Model

Sensor:

σ_m = 100 m

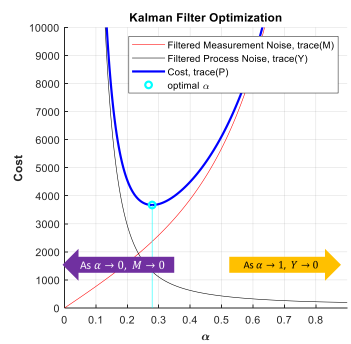
Δt = 0.5 sec

Filter:

q = 10 m/s$^{3/2}$

Optimal α =0.233

Illustration of Optimization
Kalman Filter Optimization – B&B



B&B Filter Model

Sensor:

σ_m = 100 m

Δt = 0.5 sec

Filter:

σ_v = 10 m/s

Optimal α =0.279

Is the Filter Truly Optimized?

- Criterion for Optimization of the Kalman filter:
  - $Trace(P) = \sigma_p^2 + \sigma_v^2$
- Assumptions:
  - Measurement noise is zero mean, Gaussian
  - Target dynamic model is representative of the object to be tracked
  - Process noise model is appropriate
  - Deviations from the target dynamic model are zero mean, Gaussian
- The output to a filter is only as accurate if the assumptions in the filter are accurate
  - How well does your filter represent the system you're trying to emulate?
  - Is the filter "optimal" if the assumptions are wrong?
- Statistical analysis can indicate if the results are "appropriate" but not optimal unless the true object kinematic states are known

Simple 1-D Filter Examples

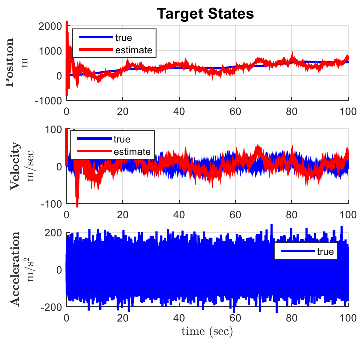
- To gain more insight into the target state estimates and its expected covariance a simple 1-D simulation is utilized
- A DWNA Kalman filter will be used to estimate an object's position and velocity states
- Let's see how the filter behaves in two separate cases
  - Case 1: Target has DWNA to match the filter model
  - Case 2: Target applies a constant acceleration for a portion of the simulation

Case 1: Target States and Estimates

Target accelerates continually. Every 0.05 seconds, a new random draw is taken, providing a constant acceleration until the next random draw. The target acceleration the following characteristics: $\mu = 0, \sigma = 5G\ (49.05\frac{m}{s^2})$

Target filter is assumes DWNA error

The target filter does not estimate acceleration



Sensor:

σ_m = 1000 m

Δt = 0.05 sec

Filter:

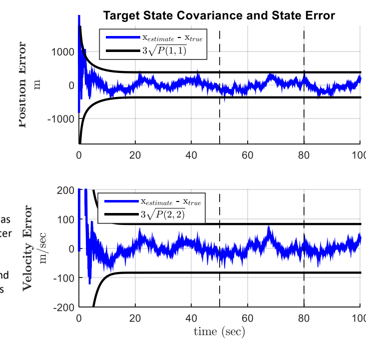
σ_w = 49.05 m/s²

τ = 6.3606 sec

Case 1: Target Covariance

The estimation errors are contained within the 3σ bounds of the covariance matrix, P, throughout the target trajectory.

This is hardly unexpected as the assumptions in the filter match the true target dynamics as well as the measurement accuracy and update rate characteristics



Sensor:

σ_m = 1000 m

Δt = 0.05 sec

Filter:

σ_w = 49.05 m/s²

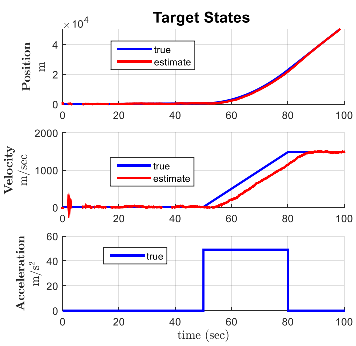
τ = 6.3606 sec

Case 2: Target States and Estimates

Target accelerates linearly at 5G for 30 seconds beginning at a time of 50 seconds. Prior this acceleration, net acceleration on the target is zero.

Target filter is assumes DWNA error

The target filter does not estimate acceleration



Sensor:

σ_m = 1000 m

Δt = 0.05 sec

Filter:

σ_w = 49.05 m/s²

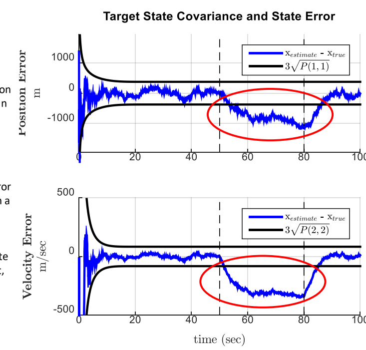
τ = 6.3606 sec

Case 2: Target Covariance

Prior to the target acceleration, the estimation errors are contained within the 3σ bounds of the covariance matrix, P (and also matrix M – not shown)

Incorrect target model error assumptions may result in a bias in the target filter estimates

When the unmodeled state assumptions are incorrect, the covariance may not properly represent the empirical data



Sensor:

σ_m = 1000 m

Δt = 0.05 sec

Filter:

σ_w = 49.05 m/s²

τ = 6.3606 sec

Recap of Case 1 and Case 2

- For a well behaved target (model assumptions are similar to the true target characteristics), the filter is a powerful tool to predict both state estimates AND the errors in the states estimates (covariance)
- Target covariance estimates do not accurately describe the estimate accuracy if the target model assumptions are incorrect in either magnitude or type of model error
  - Example of incorrect scale of process noise
    - Process noise in the model is less than the true process noise
    - Process noise, $\sigma_w = 5G$, but the actual standard deviation is 10G
  - Example of erroneous modeling type
    - Even though both the target and σ_w were equal to 5G, the process noise model did not match how the target actually behaved
    - Our example assumed DWNA but the process noise was not "noise" at all!
- Is there a way to predict the maximum bias if the maximum expected value of unmodeled state is known?

Maximum Bias Can Be Estimate as a Worst Case Error By Using Derived Filter Characteristics

Derived Filter Characteristics

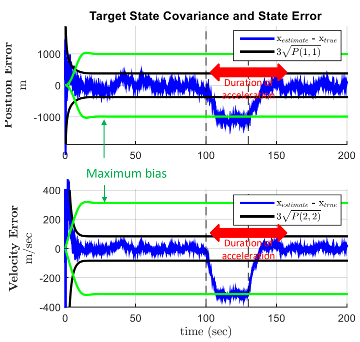
- Derived filter characteristics are not readily available from the Kalman filter equations, but can be computed from the filter characteristics such as filter gains
- Filter responsiveness
  - The quickness in which the filter reacts to change is characterized by the filter response time constant, τ
  - For $\alpha - \beta$ filters, $\tau = \left(\frac{2\alpha - \beta}{2\beta}\right)\Delta t$
- Filter lag
  - Due to the nature of the filter (reacting to observations), the response of the filter to a change in target model characteristics will lag the truth
  - Filter lag, when describing the effect of a un-modeled state on the filter estimate, is called bias
  - For the $\alpha - \beta$ filter, the bias terms for position and velocity for a given acceleration is
    - $Bias_p = \frac{1-\alpha}{\beta}\Delta t^2\ Acc = \frac{1}{2}\tau^2\ Acc$
    - $Bias_v = \left(\frac{2\alpha-\beta}{2\beta}\right)\Delta t\ Acc = \tau\ Acc$

Case 2: Filter τ and Target State Bias

One can see the bias that exists in the target position and velocity estimates due to the unmodeled acceleration

Assuming a maximum deterministic maneuver of 5G provides a bounds

The bias bound does not consider the random error, hence the estimate errors are slightly larger than the bias estimate



Sensor:

σ_m = 1000 m

Δt = 0.05 sec

Filter:

σ_w = 49.05 m/s²

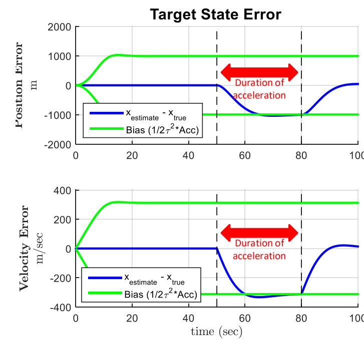
τ = 6.3606 sec

Case 2: Worst Case Bias

To determine the magnitude of the bias alone, one can set each measurement to the true position, but report σ_m^2 as normal.

One can see the bias that exists in the target position and velocity estimates due to the unmodeled acceleration

The sluggishness of the filter to respond to the change in trajectory due to acceleration is evident, as it takes roughly 3τ seconds for the filter to reach a new steady state whenever an acceleration change occurs



Sensor:

σ_m = 1000 m

Δt = 0.05 sec

Filter:

σ_w = 49.05 m/s²

τ = 6.3606 sec

Target Bias & Worst Case Uncertainty

- When designing a system, it is often beneficial to understand "how wrong" the state estimates of a track could be
- The benefit of determining the bias assuming the worst case target model error is that the error can be bound and system level analysis can be based upon magnitude of these errors
  - The computation of a bias to bound a problem space is common
  - A Kalman filter is not needed to compute the maximum bias estimates, only the filter gains and the measurement update period

The Filtering Debates

- There are some basic philosophical differences when discussing recursive filters which are unresolvable
  - Often resulting in discussions of whose filter is "most correct"
  - Many filter offshoots have been developed to "better" model the real world
- The majority of the arguments can be traced back to the inability of mathematical equations to perfectly simulate nature
  - Dr. Kalman himself stated that the optimality of any state estimator is suspect [4,5]
  - It is a tool in which one attempts to emulate nature through mathematical equations
  - Nature doesn't conform to the equations which man developed
- As the error in each assumption made is reduced, a more complex filter model is born
  - $\alpha - \beta$ filter
  - Kalman filter (and its offshoots)
  - Single target model assumption filter
  - Multiple target model assumption filter
    - Interacting filters
    - Particle filters

Filter Complexity

A Sample of the Kalman Filter Family

The Filtering Debates

- Schmidt-Kalman Filter (SKF)
  - Developed to filter data when a measurement is known to contain a bias
  - Description (covariance) of the bias is required for proper usage
- Extended Kalman Filter (EKF)
  - Linearizes all equations of motion of measurement definitions
  - This model is often preferred for tracking radars with large cross-range errors and the linear approximation of $\sigma_m = R\,\sigma_a$ is invalid or when target motion model is non-linear
  - The model assumes the error incurred in a linear state propagation is small
- Unscented Kalman Filter (UKF)
  - Uses n points (called sigma points) from previous measurement events to linearize the non-linear random variable function
  - The linear regression technique is more accurate than the approximations used in the EKF but requires the selection of sigma points to compute the correction terms to the non-linear equations

Suggested Reference Material

- Good Reference Material:
  1. Bar-Shalom, *Estimation with Applications to Tracking and Navigation: Theory Algorithms and Software*. 2001
  2. Gelb, *Applied Optimal Estimation*. MIT Press, 1974 (out of print, but easy to find in the library)
- Canonical Track Filtering Documents
  3. Kalman, R.E., *A New Approach to Linear Filtering and Prediction Problems*. ASME-Journal of Basic Engineering, 82 (Series D): 35-45. 1960.
  4. Kalman, R.E., *Randomness Reexamined*. Modeling, Identification, and Control. Vol 15, No. 3., 141-151. 1994.
  5. Kalman, R.E., *Discovery and Invention: The Newtonian Revolution in System Technology*. Journal of Guidance, Control, and Dynamics, Vol 26, No. 6, November-December 2003.
  6. Benedict & Bordner, *Synthesis of an Optimal Set of Radar Track-Wile-Scan Smoothing Equations*. IRE Transactions on Automatic Control. 1962